Horizon Boundary Mechanics: Force, Pressure, and Work from the Einstein Boundary Potential

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La Jolla, California, USA; revised 14 November 2026

Abstract

Recognizing acceleration length as a geometrically normalized work coordinate of the established Einstein boundary potential identifies a generalized boundary force and normal-scale pressure. Their action-supported work interpretation joins the area-conjugate surface stress in one macroscopic state law. The selected nonexpanding-null action response supplies the principal canonical justification; the stationary area–boost route gives a complementary interpretation with shared clock and area normalization. Schwarzschild work becomes $M\kappa\, dR = P_R\, dV$, with $P_R = 2P_L$. A finite Schwarzschild cavity realizes the length variation through a physical wall clock and relates it to Brown–York thermodynamics. The separately assigned instantaneous equilibrium FRW screen has $P_R = P_L = \rho/3$ and potential equal to the Misner–Sharp horizon energy. Direct null-boundary antecedents and earlier cosmological equations of state are distinguished from the added length-work interpretation. Temperature and entropy enter after the mechanics.

Keywords: horizon boundary mechanics, gravitational boundary response, acceleration length, null symplectic structure, FRW cosmology

1. Geometry and the boundary potential

Black-hole mechanics fixes an area coefficient before its thermal interpretation [1]. Membrane stresses and cosmological horizon thermodynamics also have established mechanical descriptions [3, 25, 9]. The question considered here is which geometrically normalized normal displacement pairs with the rate response of the known Einstein boundary potential, and how that response relates to horizon-size work.

A horizon cut has transverse area A and a surface-gravity scale κ, with its physical time normalization specified. Express the same acceleration scale as a length:

$$L_s = \frac{c^2}{\kappa}, \qquad L_\kappa = |L_s|, \qquad \kappa \neq 0. \tag{1}$$

In Rindler geometry L_κ is the normal observer–horizon proper distance on the instantaneous rest slice. The characteristic time $t_\kappa = c/|\kappa|$ gives $L_\kappa = ct_\kappa$ and, in stationary normal flow, one unit of boost rapidity. Neither a photon transit from the horizon nor a universal spherical radius is assumed.

For the Einstein boundary prescription used here, the energy-valued potential is [5, 6, 7, 11, 12]

$$\boxed{U_\partial(A, L_s) = \frac{A\kappa c^2}{8\pi G} = \frac{c^4}{8\pi G}\frac{A}{L_s}.} \tag{2}$$

The Einstein boundary structure fixes the normalization (Sec. 3), not dimensional analysis. With $\kappa_E = 8\pi G/c^4$, $1/\kappa_E$ has energy-per-length units and A/L_s supplies the length factor.

What displacement pairs with this potential's work? Its L_s dependence identifies a horizon-scale force alongside the area response. Here $U_\partial = Mc^2/2$ for Schwarzschild and $U_\partial = E_{\mathrm{MS}}$ for the

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equilibrium-FRW screen; Table 1 separates their inputs and support.

2. One differential, common mechanical responses

At fixed c and G, differentiating before imposing a particular horizon family's relation between A and L_s gives

$$\begin{aligned} dU_\partial &= \frac{c^4}{8\pi G L_s}\,dA - \frac{Ac^4}{8\pi G L_s^2}\,dL_s \\ &= \Pi_\partial\,dA - F_L\,dL_s. \end{aligned} \tag{3}$$

Writing $M_\partial = U_\partial/c^2$ for the potential's mass equivalent, the coefficients are

$$\Pi_\partial = \left(\frac{\partial U_\partial}{\partial A}\right)_{L_s} = \frac{U_\partial}{A} = \frac{\kappa c^2}{8\pi G}, \tag{4}$$

$$F_L = -\left(\frac{\partial U_\partial}{\partial L_s}\right)_A = \frac{U_\partial}{L_s} = M_\partial \kappa. \tag{5}$$

The established area stress and the length-conjugate force are derivatives of the same potential. The normal-scale pressure is

$$\boxed{P_L = \frac{F_L}{A} = \frac{\Pi_\partial}{L_s} = \frac{c^4}{8\pi G L_s^2} = \frac{\kappa^2}{8\pi G}.} \tag{6}$$

Pressure is a response density, not a third canonical pair; $F_L = M_\partial\kappa$ is an identity, not an additional material equation of motion.

The integrated energy identity is

$$U_\partial = \Pi_\partial A = F_L L_s = P_L A L_s. \tag{7}$$

Since $U_\partial = A\Pi_\partial$, $A\,d\Pi_\partial = -F_L\,dL_s$ and $d\Pi_\partial = -P_L\,dL_s$. Thus pressure is the negative stress slope across boundary configurations, not a material-layer gradient. The scale laws

$$\frac{1}{L_s} = \kappa_E \Pi_\partial, \qquad \frac{1}{L_s^2} = \kappa_E P_L \tag{8}$$

are consequences of the common mechanics, not its starting hierarchy.

No enclosed volume is needed. If $\vartheta_V = A\,dL_s$,

$$F_L\,dL_s = P_L \vartheta_V, \qquad d\vartheta_V = dA \wedge dL_s. \tag{9}$$

Thus ϑ_V is not a global volume differential on the independent source space. The unchanged P_L multiplies dV only when $dV = A\,dL_s$. A sphere instead has $dV = A\,dR$ and the transformed pressure below; AL_s is not generally its volume.

3. Canonical foundation of the response

The action variation gives interior field equations and a boundary one-form Θ, linear in a field variation like $p\,\delta q$. Its field-space derivative is the canonical two-form. Boundary terms select its source and response; Θ is distinct from the energy U_∂.

Let a null hypersurface $\mathcal{N}$ be foliated by spacelike two-dimensional cuts S, with metric q_{AB} and area element $\epsilon_S = \sqrt{q}\,d^2x$. For a selected null generator ξ^a, define

$$\nabla_\xi \xi^a = \widehat{\kappa}\xi^a, \qquad \theta = \frac{1}{\sqrt{q}}\mathcal{L}_\xi \sqrt{q}. \tag{10}$$

Inaffinity $\widehat{\kappa}$ measures nonaffine parameterization; θ measures fractional area change. Both have inverse-length units, with $\kappa = c^2\widehat{\kappa}$. A null generator has no proper time or material proper acceleration.

3.1. Established functional and the selected source

The direct antecedent is the null-boundary functional of Parattu, Chakraborty, Majhi and Padmanabhan [6]. With the orientation used here and $c = 1$,

$$B_0 = \frac{1}{8\pi G}\int_{\mathcal{N}} (\theta + \widehat{\kappa})\epsilon_S\,dv. \tag{11}$$

Chakraborty and Padmanabhan [7], Eqs. (11)–(12), identify its rate term as integrated heat content and its expansion term as $(A_f - A_i)/(8\pi G)$. For uniform stationary nonexpanding data the rate term is $U_\partial \Delta\tau$. Thus neither the functional nor the stationary TS value is introduced here. Parattu, Majhi and Padmanabhan [5] already display the complementary $T\delta S$ and $S\,\delta T$ responses. The additional construction selects a controlled rate response and identifies its geometrically normalized length coordinate and force.

Hopfmüller and Freidel identify the scalar canonical combination [11, 12]

$$\mu = \widehat{\kappa} + \frac{1}{2}\theta. \tag{12}$$

Table 1: Claim-to-evidence map. Family restrictions follow the independent-coordinate derivatives.

Claim	Inputs or prescription	Derivation and support
Force and pressure	Specified Einstein potential; independent (A, L_s); fixed c, G.	Eqs. (2)–(9): conditional algebra.
Canonical length response	Selected rate action; fixed clock, cuts and interval; $\theta = \delta\theta = 0$; controlled other responses.	Eqs. (10)–(18); full Appendix I: restricted action derivation.
Geometric length	Uniformly accelerated Rindler observer, orthogonal rest slice, independent planar area.	Sec. 5.1; full Sec. V A: geometric normalization.
Schwarzschild radius work	Infinity-normalized clock; $L_\kappa = 2R_H$; $U_\partial = Mc^2/2$; $dV_H = A_H dR_H$.	Sec. 5.2, Eqs. (28)–(29): exact family identities.
FRW screen identities	Einstein–Misner–Sharp relation $E_A = \rho V_A$ and separate CK rate c^2/R_A.	Eqs. (36)–(44): assigned screen-state identities.
Thermal interpretation	Hawking–Unruh/KMS normalization and Einstein entropy with zero additive constant.	Eq. (24), Sec. 6: additional physical dictionary.

Shear and generator/twist data occupy other sectors. In units $c = 1$, the area polarization has scalar term $\Theta_A^{(0)} = -(8\pi G)^{-1}\int \mu\,\delta\epsilon_S\,dv$. Adding the exact variation of $B_\mu = (8\pi G)^{-1}\int \mu\epsilon_S\,dv$ gives

$$\Theta_\mu^{(0)} = \Theta_A^{(0)} + \delta B_\mu = \frac{1}{8\pi G}\int_{\mathcal{N}} \epsilon_S\,\delta\mu\,dv. \tag{13}$$

This polarization exchange, $-p\delta q + \delta(pq) = q\delta p$, leaves the two-form unchanged. Covariant mixed conditions identify the same derivative-containing source [13]; Table 2 specifies the correspondence.

Restrict to the nonexpanding sector and variations tangent to it:

$$\theta = 0, \qquad \delta\theta = 0. \tag{14}$$

Restoring acceleration units and the prescribed physical time label τ gives

$$\Theta_\kappa^{(0)} = \frac{c^2}{8\pi G}\int d\tau \int_S \epsilon_S\,\delta\kappa. \tag{15}$$

The coordinate change follows directly from $L_s = c^2/\kappa$: $\delta\kappa = -c^2\delta L_s/L_s^2$. Substituting it into the gravitational response gives

$$\boxed{\begin{aligned}\Theta_L^{(0)} &= -\int d\tau \int_S \epsilon_S\,\frac{c^4}{8\pi G L_s^2}\,\delta L_s \\ &= -\int d\tau \int_S \epsilon_S\,P_L\,\delta L_s.\end{aligned}} \tag{16}$$

For a common length variation, $F_L = \int_S P_L\,dA$; a uniform finite cut has $F_L = AP_L$ and $\Theta_L^{(0)} = -\int d\tau F_L \delta L_s$. The potential-derived coefficient is therefore the selected action response. Pressure remains a density with its area measure, not an additional independent partner of L_s.

Equivalently, $\Theta_\kappa^{(0)}/\Delta\tau = A\delta\Pi_\partial$ for uniform stationary data. The exchange $-\Pi_\partial\delta A + \delta(A\Pi_\partial) = A\delta\Pi_\partial$ describes the same structure; neither selected one-form alone equals dU_∂.

Source variations compare nearby boundary-value problems with controlled other data, preserving nullness and tangent nonexpansion. Within each conservative problem the mixed source is fixed. The same clock prescription permits $\delta\kappa \neq 0$ as the geometry changes.

Equivalently, the mixed on-shell action has local Hamilton–Jacobi response

$$F_L(\tau) = -\frac{\delta I_{\mathrm{os}}}{\delta L_s(\tau)}. \tag{17}$$

The time measure makes this a force; a constant-source derivative over a fixed interval is instead $-\Delta\tau F_L$. The action and counterterm prescription matter. Arbitrary source-dependent additions can shift the response, and arbitrary-source existence is not asserted.

In this scalar sector,

$$\frac{\mathbf{\Omega}^{(0)}}{\Delta\tau} = \delta A \wedge \delta\Pi_\partial = \delta L_s \wedge \delta F_L. \tag{18}$$

These are one pair; the Legendre term selects rate polarization. Fix $A > 0$, c, G and the clock for the rate variation. An invertible rate-only coordinate

$q(\kappa)$ preserving the area source, on a connected $\kappa \neq 0$ branch, satisfies

$$\begin{aligned} -F_q\, dq &= \frac{Ac^2}{8\pi G}\, d\kappa, \qquad F_q = M_\partial \kappa \\ \Longrightarrow \quad \frac{dq}{d\kappa} &= -\frac{c^2}{\kappa^2}, \qquad q = L_s + q_0. \end{aligned} \tag{19}$$

The additional product $U_\partial = F_q q$ fixes $q_0 = 0$, agreeing with Rindler distance. This is conditional uniqueness: other coordinates remain canonical with transformed forces.

Companion Appendix I derives the current from Einstein–Hilbert metric jets for the selected action. Equations (11) and (12) give, in $c = 1$ units,

$$\boxed{I_{\text{rate}} = I_{\text{EH}} - B_0 + B_\mu = I_{\text{EH}} - \frac{A_f - A_i}{16\pi G}.} \tag{20}$$

The half-area endpoint follows from $\mu = \widehat{\kappa} + \theta/2$, and differs from the full expansion endpoint in B_0 alone. Tangent nonexpansion gives $\delta A_f = \delta A_i$, so this endpoint has vanishing allowed variation. The Einstein–Hilbert shorthand is therefore justified only with these endpoint and variation conditions attached.

The complementary area prescription removes the explicit rate coefficient but retains the scalar two-form. This source dependence is part of the variational problem, not evidence for a source-independent force. Similarly, an affine generator sets a segment's inaffinity to zero without by itself removing the complete null-and-joint contribution: Lehner et al. show its reparameterization behavior and the nonexpanding cancellation [8]. Prescribed clock, cuts, orientation and other-response conditions remain essential. A rescaling of the physical clock changes the specified source; auxiliary-normal rescalings that preserve that clock do not.

3.2. Stationary area–boost interpretation

A corner carries an area–boost pair [14, 15]. With Eq. (16)'s orientation and the same stationary clock, its action-valued charge and boost rate give

$$\begin{aligned} J_\eta &= \frac{Ac^3}{8\pi G}, \qquad \omega_s = \frac{\kappa}{c} = \frac{c}{L_s}, \\ &\boxed{U_\partial = J_\eta \omega_s, \qquad U_\partial L_s = c J_\eta.} \end{aligned} \tag{21}$$

The boost changes the time–normal frame. At fixed stationary duration $\Delta\tau$, $\eta = c\Delta\tau/L_s$ gives

$$\frac{J_\eta \delta\eta}{\Delta\tau} = -\frac{cJ_\eta}{L_s^2}\delta L_s = -F_L \delta L_s. \tag{22}$$

This complementary recovery requires the shared clock, area normalization and stationary boost relation; it is not a complete Hamiltonian variation or an arbitrary cut's evolution law.

3.3. Scope of the action result

Stationary Rindler, Schwarzschild and de Sitter realize the same selected response. Established de Sitter action thermodynamics [29, 30] supplies an equilibrium FRW reference: choose $R_0 = R_A(t_0)$, $\Lambda_0 = 3/R_0^2$ and central proper time. Its positive-magnitude $\kappa_0 = c^2/R_0$, $L_0 = R_0$ reproduce the CK values. The companion's stationary de Sitter discussion gives the metric and action.

At fixed Λ_0, the cosmological term adds no derivative boundary response. Along a varying-radius family, $\delta\Lambda_0 = -6\delta R_0/R_0^3$ requires the bulk coupling variation; R_0 is constant within one de Sitter solution. An actual FRW horizon's motion remains separate. Vanishing null expansion on each cut does not make its worldtube nonexpanding or null; expanding-null sources retain μ and the other responses.

The varying-Λ comparison has an established extended-thermodynamic form [31]. On the pure de Sitter family,

$$\begin{aligned} P_\Lambda &= -\frac{\Lambda c^4}{8\pi G} = -3P_{\text{scr}}, \\ T\, dS &= V_A\, dP_\Lambda = -3V_A\, dP_{\text{scr}}. \end{aligned} \tag{23}$$

This is the cosmological-horizon first law with vanishing mass parameter and varying coupling; its zero mass charge is not E_{MS}. It describes a family of solutions, not matter crossing one fixed-Λ de Sitter horizon.

4. Boundary potential versus mass first law

The mass first law is a Hamiltonian variation for a specified evolution, including the symplectic-potential subtraction and any generator corrections; it is not the full differential of $A\kappa$. Thus

Table 2: Closest canonical antecedents and the added interpretation. Source units and normalizations are matched first.

Earlier source and equations	Established foundation adopted	Length-coordinate interpretation here
Parattu et al. [5], (111)–(113).	Static area-weighted rate response $A\delta\kappa/(8\pi G) = S\,\delta T$ ($c = 1$).	$L_s = c^2/\kappa$ gives $-F_L\delta L_s$ and $P_L = F_L/A$; Eqs. (6), (16).
Parattu et al. [6]; Chakraborty–Padmanabhan [7], (11)–(12).	Null functional $(8\pi G)^{-1}\int(\theta+\widehat{\kappa})\epsilon_S\,dv$; integrated heat content plus endpoint area.	Identify the inherited functional, selected polarization and endpoint shift before deriving its length response; Eqs. (11), (20).
Hopfmüller–Freidel [11], (1.1); [12], (1.3)–(1.4).	Area/scalar pair, $\mu = \widehat{\kappa} + \theta/2$, and the rate potential $\epsilon_{\mathcal{N}}\delta\mu$.	Tangent nonexpansion and the area measure give the selected length response, Eqs. (13)–(18).
Odak et al. [13], (3.26)–(3.28), (4.6), (4.8).	Scalar polarizations, mixed source $\theta + 2k = 2\mu$ ($k = \widehat{\kappa}$: null inaffinity), and a separate corner condition.	Match the scalar source; our selected action and endpoints are specified in full Appendix I, not identified with every mixed problem.

$d(TS) = T\,dS + S\,dT$ does not add $S\,dT$ to that law. Surface-Hamiltonian treatments already give the complementary terms [5, 4]; here the rate term acquires the length-force and pressure interpretation (Table 2). Thermally, the source exchange is $-T\delta S + \delta(TS) = S\,\delta T$.

Where the positive-branch thermal identification applies, differentiating $U_\partial = \omega_s J_\eta$ gives

$$\begin{aligned} \omega_s\,dJ_\eta &= \Pi_\partial\,dA = T\,dS, \\ J_\eta\,d\omega_s &= -F_L\,dL_\kappa = S\,dT. \end{aligned} \tag{24}$$

The scale leg has exterior derivative $(c^2/8\pi G)\delta A \wedge \delta\kappa$, whereas dU_∂ is exact. Its integrability along a one-dimensional family does not make it an independent state energy.

For Schwarzschild, $E = Mc^2 = 2U_\partial$ and the constrained family gives

$$dE = T\,dS = 2F_L\,dL_\kappa, \qquad dU_\partial = F_L\,dL_\kappa. \tag{25}$$

Appending the boundary scale term as an independent addition to $dE = T\,dS$ would double-count that variation.

Kerr–Newman likewise has $F_L\,dL_\kappa = -S\,dT$ constrained by its usual first law and Smarr relation [1, 2]; the companion gives the explicit complement. At finite area, $\kappa \to 0^+$ sends $U_\partial, F_L, P_L \to 0$ and $L_\kappa \to \infty$; the κ response remains regular.

5. Stationary and cosmological realizations

Table 1 separates stationary realizations from assigned cosmological screens. Static de Sitter uses positive-magnitude $L = R_A$. On a spherical marginal horizon, $A = 4\pi R^2$ and the Misner–Sharp energy is $E_{\rm MS} = c^4R/(2G)$ [24, 25]. The area potential therefore has the spherical form

$$\boxed{U_\partial = E_{\rm MS}\frac{R}{L_s}, \qquad U_\partial L_s = cJ_\eta = E_{\rm MS}R.}$$

For $\kappa > 0$, $U_\partial/E_{\rm MS} = R/L_\kappa$ is one in Cai–Kim FRW and one half in Schwarzschild. This realizes the area potential on a marginal horizon, not an arbitrary sphere; differentials retain $E_{\rm MS} \propto R$.

5.1. Rindler: the acceleration-length interpretation

For constant proper acceleration a in Minkowski spacetime, $\kappa = a$, $L_\kappa = c^2/a$ and $P_L = a^2/(8\pi G)$. The normal rest-slice distance is c^2/a; planar area is independent. Frodden–Ghosh–Perez [16] obtain $E_{\rm loc} = Ac^4/(8\pi G\ell)$ at leading order for local observers at proper horizon distance ℓ, with $\ell^2 \ll A$. Their local clock differs from the infinity-normalized Schwarzschild prescription; its $L_\kappa = 2R_H$ is not a near-horizon proper distance.

5.2. Schwarzschild: work of horizon-size change

With stationary Killing time normalized at infinity and areal radius R_H defined by $A_H = 4\pi R_H^2$,

$$\kappa = \frac{c^2}{2R_H}, \quad L_\kappa = 2R_H, \quad F_L = \frac{c^4}{8G}, \quad U_\partial = \frac{Mc^2}{2}. \tag{26}$$

The Smarr relation becomes

$$\boxed{Mc^2 = TS + F_L L_\kappa, \qquad TS = F_L L_\kappa = \frac{Mc^2}{2}.} \tag{27}$$

The products coincide pointwise and represent different derivative directions of one potential, not independently stored energies. The same work expressed through the actual horizon size is

$$\boxed{\begin{aligned} F_R &= 2F_L = \frac{U_\partial}{R_H} = M\kappa, \\ P_R &= \frac{F_R}{A_H} = 2P_L = \frac{\kappa^2}{4\pi G}, \\ F_L\, dL_\kappa &= F_R\, dR_H = P_R\, dV_H. \end{aligned}} \tag{28}$$

Here $V_H = 4\pi R_H^3/3$ and $dV_H = A_H\, dR_H$. Since $dL_\kappa/L_\kappa = dR_H/R_H$, length work measures the same fractional horizon-size change, with no displacement of another surface at $2R_H$. The areal radius is a geometric measure of size, not a universal proper distance. The complete law is

$$\boxed{\begin{aligned} dU_\partial &= \Pi_\partial\, dA_H - M\kappa\, dR_H \\ &= T\, dS - P_R\, dV_H. \end{aligned}} \tag{29}$$

Area and volume co-vary along this family: $\Pi_\partial\, dA_H = 2P_R\, dV_H$, $dU_\partial = P_R\, dV_H$, and $U_\partial = 3P_R V_H$. The mass law remains Eq. (25). Transforming the displacement and its coefficient preserves the selected canonical response of Sec. 3; it adds no independent source variation.

5.3. A finite cavity and a physical wall clock

A Schwarzschild cavity gives a concrete stationary realization of the length variation within four-dimensional Einstein gravity [17, 18]. Put the wall at $r_B > R_H$, set $N_B = \sqrt{1 - R_H/r_B}$ and normalize Killing time to the wall's proper time. The horizon area $A_H = 4\pi R_H^2$ and wall area $A_B = 4\pi r_B^2$ are distinct. The horizon quantities referred to this clock are

$$\begin{aligned} \kappa_B &= \frac{c^2}{2R_H N_B}, \qquad L_B = 2R_H N_B, \\ U_B &= T_B S = \frac{c^4 R_H}{4GN_B} = \frac{A_H c^4}{8\pi G L_B}, \\ F_L^B &= \frac{U_B}{L_B} = \frac{c^4}{8GN_B^2}. \end{aligned} \tag{30}$$

The pressure $F_L^B/A_H = \kappa_B^2/(8\pi G)$ uses the horizon area. With the standard flat reference subtraction, the Brown–York energy and surface pressure are

$$\begin{aligned} E_{\rm BY} &= \frac{c^4 r_B}{G}(1 - N_B), \\ p_{\rm BY} &= \frac{c^4 (1 - N_B)^2}{16\pi G r_B N_B}. \end{aligned} \tag{31}$$

Here $p_{\rm BY}$ has force-per-length units. Their established first law and quasilocal Euler relation [19] are

$$\begin{aligned} dE_{\rm BY} &= T_B\, dS - p_{\rm BY}\, dA_B, \\ U_B &= \frac{1}{2} E_{\rm BY} + p_{\rm BY} A_B. \end{aligned} \tag{32}$$

At fixed horizon area, $dS = 0$ and $dU_B = -F_L^B\, dL_B$. Differentiating the second relation and using the first gives

$$\boxed{F_L^B\, dL_B = -\frac{1}{2} p_{\rm BY}\, dA_B - A_B\, dp_{\rm BY}.} \tag{33}$$

The wall radius varies L_B without changing A_H:

$$\left(\frac{\partial L_B}{\partial r_B}\right)_{R_H} = \frac{R_H^2}{r_B^2 N_B} > 0. \tag{34}$$

Hence (A_H, L_B) are independent local coordinates on the finite two-parameter cavity family, with $0 < L_B < 2R_H$. This supplies a physical clock prescription and a stationary realization of the scale response. Moving the wall changes the physical generator $\xi_B = \xi_\infty/N_B$. Identifying this family with the strictly fixed-generator null-action ensemble requires a separate treatment of that field dependence, boundary transport and joints.

It does not identify that response with the net force on the wall. The two energies in Eq. (32) have

different work coefficients; at fixed R_H,

$$\frac{F_L^B\, dL_B}{p_{\rm BY}\, dA_B} = \frac{(1+N_B)^2}{4N_B^2}. \tag{35}$$

Moreover, the wall observer's proper acceleration is $a_B = c^2 R_H/(2r_B^2 N_B)$, so $a_B/\kappa_B = (R_H/r_B)^2$. The exact identity $F_L^B = (U_B/c^2)\kappa_B$ therefore involves wall-normalized *horizon surface gravity*, not generally the local weight $(U_B/c^2)a_B$. Near the horizon the accelerations agree to leading order. At infinity $N_B \to 1$ recovers the Schwarzschild normalization above. A claim about extracted work delivered to a recipient would additionally require a wall-and-actuator process and its energy accounting.

5.4. Cosmological application: the FRW screen

For four-dimensional FRW with arbitrary spatial curvature, $R_A = c/\sqrt{H^2 + kc^2/a^2}$, where $H = \dot{a}/a$. The geometric Misner–Sharp energy at a marginal sphere and its Einstein–FRW matter identification give [24, 33]

$$E_A = \frac{c^4 R_A}{2G} = \rho V_A, \qquad V_A = \frac{4\pi R_A^3}{3}. \tag{36}$$

Here ρ, p_m are total bulk energy density and pressure in the same units, including vacuum. Areal volume V_A is not generally proper spatial volume for $k \neq 0$. The equality $E_A = \rho V_A$ is the Einstein matter–geometry input, not a result of defining pressure.

The equilibrium FRW (instantaneous Cai–Kim) prescription [20] sets

$$\kappa_{\rm CK} = \frac{c^2}{R_A}, \qquad L_{\rm CK} = R_A. \tag{37}$$

Then the force, screen pressure, and integrated identity are

$$F_L^{\rm CK} = \frac{c^4}{2G}, \quad P_{\rm scr} = P_L^{\rm CK} = \frac{c^4}{8\pi G R_A^2} = \frac{\rho}{3}, \tag{38}$$

$$E_A = \rho V_A = U_\partial = TS = F_L L_{\rm CK} = 3P_{\rm scr} V_A. \tag{39}$$

Here $L_{\rm CK} = R_A$ gives $F_R = F_L = M_A\kappa_{\rm CK}$ and $P_R = P_L = P_{\rm scr}$, with $M_A = E_A/c^2$: acceleration-length and radius work coincide. The thermal equality uses the CK temperature and Einstein entropy. The complete constrained differential is

$$dE_A = \Pi_{\rm CK}\, dA_A - P_{\rm scr}\, dV_A = T_{\rm CK}\, dS_A - P_{\rm scr}\, dV_A. \tag{40}$$

Along the spherical family, $dE_A/dV_A = P_{\rm scr}$. This total slope differs from the unconstrained derivative $-(\partial E/\partial V)_S$ and from cosmic material pressure.

Equation (40) is a screen-state law. Using $\rho = 3P_{\rm scr}$ and bulk conservation, Cai–Kim's matter-supply heat [20, 21], with their crossing orientation, becomes

$$\begin{aligned}\delta Q_{\rm CK} &= A_A(3P_{\rm scr} + p_m)HR_A\, dt \\ &= -3V_A\, dP_{\rm scr} = T_{\rm CK}\, dS_A = 2\, dE_A.\end{aligned} \tag{41}$$

The moving sphere obeys $dE_A = 3P_{\rm scr}\, dV_A - \delta Q_{\rm CK}$, consistent with Eq. (40). Crossing flux and total energy change therefore differ. These identities do not establish exact equilibrium thermality during arbitrary FRW evolution.

With $\kappa_E = 8\pi G/c^4$, the pressure representation is

$$\frac{1}{R_A^2} = \frac{H^2}{c^2} + \frac{k}{a^2} = \kappa_E P_{\rm scr}, \tag{42}$$

$${}^{(2)}\mathcal{R}_\perp \equiv \frac{2\ddot{a}}{ac^2} = -\kappa_E(P_{\rm scr} + p_m), \tag{43}$$

$$\dot{P}_{\rm scr} + H(3P_{\rm scr} + p_m) = 0. \tag{44}$$

$1/R_A^2$ is the horizon sphere's Gaussian curvature; ${}^{(2)}\mathcal{R}_\perp$ is the Ricci scalar of the normal time–radial metric, not the full spacetime Ricci scalar. Equation (44) gives $p_m = -3P_{\rm scr} - \dot{P}_{\rm scr}/H$ for $H \neq 0$: screen pressure fixes density, while its evolution retains matter-pressure information. These represent the same Friedmann dynamics for any spatial curvature.

The signed Hayward cross-focusing rate is different [25, 26]:

$$\frac{\kappa_H^s}{\kappa_{\rm CK}} = \frac{p_m - P_{\rm scr}}{4P_{\rm scr}} = \frac{3w-1}{4}, \qquad w = p_m/\rho. \tag{45}$$

In radiation FRW it vanishes while $P_{\rm scr}$ remains finite; in de Sitter their magnitudes agree. The associated quadratic response is $P_L^H = (P_{\rm scr} - p_m)^2/(16P_{\rm scr})$. On a nonzero branch its scale work

obeys

$$F_L^H\, dL_H^s = \frac{p_m - P_{\rm scr}}{4}\, dV_A - \frac{3}{4} E_A\, dw. \tag{46}$$

The volume coefficient at fixed w is $P_R^H = (p_m - P_{\rm scr})/4$, with $P_L^H = (P_R^H)^2/P_{\rm scr}$. Along a varying-$w$ history the displayed dw term remains. The pointwise reconstruction $p_m = P_{\rm scr} + 4P_R^H$ requires no constant-w assumption; it uses the Einstein–FRW equations. P_R^H is signed and distinct from Hayward work density.

Earlier FRW equation of state. Hayward's work density is $W = (\rho - p_m)/2$. It is the pressure variable in earlier FRW thermodynamic equations of state [22, 23]. For $\Lambda = 0$, those formulations use internal energy $U_{\rm old} = -E_A$ and pressure $P_{\rm old} = W$, with $dU_{\rm old} = T_H\, dS - W\, dV_A$ on the negative-Hayward branch. With $\ell_P^2 = G\hbar/c^3$, their equation of state in SI units is

$$\begin{aligned} W &= \frac{k_B T_H}{2\ell_P^2 R_A} + \frac{c^4}{8\pi G R_A^2} = P_{\rm scr} - 2P_R^H, \\ T_H &= \frac{\hbar|\kappa_H^s|}{2\pi k_B c}, \qquad \kappa_H^s < 0. \end{aligned} \tag{47}$$

The sign condition is necessary for the displayed positive thermal term to equal $-2P_R^H$. The algebraic identity $W = P_{\rm scr} - 2P_R^H$ itself follows from the signed reconstruction above for total ρ, p_m, including vacuum when present. With explicit Λ, the JHEP formulation separates matter work density W_m and cosmological pressure: $P_{\rm old} = W_m - P_\Lambda = W_{\rm total}$. This agrees after converting to the total-fluid convention.

The isolated inverse-square term in the earlier equation of state is exactly $P_{\rm scr}$; its total pressure also includes the thermal term. In pure de Sitter that thermal contribution is $2P_{\rm scr}$, giving $W = 3P_{\rm scr}$. More generally $W > P_{\rm scr}$ on the stated negative-Hayward branch, not in every FRW state: for dust $W = 3P_{\rm scr}/2$, and at radiation $W = P_{\rm scr}$ while $\kappa_H^s = 0$. The thermal boundary-coordinate construction using L_H^s is singular at that zero-rate point, although the algebraic work-density identity remains meaningful. The equilibrium CK screen instead keeps $L_{\rm CK} = R_A$ and finite $P_{\rm scr}$.

These are compatible descriptions using different rates, potentials and work coefficients. The additional identification is that $P_{\rm scr}$ is the force-per-area response of the chosen CK acceleration length, whereas P_R^H is the signed radial coefficient of the Hayward-rate potential. Neither identifying W with $P_{\rm scr}$ nor equating $U_{\rm old}$ with U_∂ preserves their first laws. The present comparison credits the earlier equation of state while locating the mechanical distinction in the source and energy assignments.

Exact published overlaps. Kong's Eqs. (3.6)–(3.10) [27] give $P_{\rm eff} = -dE_A/dV_A = -P_{\rm scr}$ and $E_A = -3P_{\rm eff}V_A = 3P_{\rm scr}V_A$ in four-dimensional Einstein FRW. These match the scalar pressure magnitude and energy relation. Here the length response is defined on independent boundary sources before the spherical restriction, retaining both legs of dU_∂.

In geometrized units $G = c = 1$, Ong's Eqs. (10)–(16) [32] give the shell energy $E = M/2 = TS$; the paragraph before Sec. V explicitly gives the spherical alternative $P = 1/(16\pi R_H^2) = P_R$. These are exact overlaps. Our radius work follows from $L_\kappa = 2R_H$ and the selected action, with spherical scaling $U_\partial = 2TS - 3P_R V_H$ and no shell-thickness or maximum-force premise. The companion's "Equation-level comparison with earlier FRW pressures and Ong" gives both derivations side by side.

Dimensional scope. The focus is four-dimensional Einstein gravity. One scope check prevents an incorrect extrapolation: on an equilibrium Einstein FRW horizon in spacetime dimension D, the corresponding area entropy and CK temperature give

$$U_\partial = T_{\rm CK} S = \frac{2}{D-2} E_{\rm MS}. \tag{48}$$

Thus $U_\partial = E_{\rm MS}$ is specific to $D = 4$. Outside it, $T\, dS - dE_{\rm MS}$ is not the boundary-potential work $T\, dS - dU_\partial = F_L\, dL$. In $D = 5$ the first residual vanishes on this family while the second remains nonzero. This distinction protects the four-dimensional identification without introducing a higher-curvature or higher-dimensional program.

6. Thermal translation and entropy density

A regular Hawking–Unruh/KMS state independently supplies the thermal normalization [10, 28, 29]. Writing $\Omega_\kappa = |\kappa|/c$, Euclidean regularity gives

$$c\Delta\tau_E = 2\pi L_\kappa, \quad k_B T = \frac{\hbar\Omega_\kappa}{2\pi}, \quad \frac{S}{k_B} = \frac{2\pi J_\eta}{\hbar}. \tag{49}$$

The last identity uses Einstein entropy with the standard zero additive constant. On the positive branch,

$$dU_\partial = T\,dS - F_L\,dL_\kappa, \qquad \frac{S}{k_B} = \frac{2\pi U_\partial L_\kappa}{\hbar c}. \tag{50}$$

Thus $U_\partial L_\kappa/c$ has the established meaning of boost charge before its conversion to dimensionless entropy. The area coefficient divided by temperature gives

$$\frac{S}{A} = \frac{P_L L_\kappa}{T} = \frac{k_B}{4\ell_P^2}, \qquad \ell_P^2 = \frac{G\hbar}{c^3}. \tag{51}$$

The quarter arises from the Einstein $1/(8\pi G)$ and thermal $1/(2\pi)$ normalizations.

7. Generality, scope, and conclusion

The contribution is the action-supported length-force and normal-pressure interpretation: Rindler geometry normalizes the length, the selected rate response supplies its work coefficient, and spherical restriction gives radius work. Its central object is the boundary potential U_∂, whose differential organizes the area and acceleration-length responses. The force $F_L = (U_\partial/c^2)\kappa$ and its area density arise from that specified response; the product $F_L L_s = U_\partial$ does not create a second independently stored energy.

The potential, null functional, scalar canonical structure and published special-case coefficients are acknowledged inputs and overlaps (Table 2). Their established status supplies a recognizable variational foundation and limits the originality claim: the added content is the normalized mechanical coordinate, force and pressure interpretation, with explicit comparisons across horizon settings. No claim of a new boundary action or modified Einstein dynamics is needed. The cavity supplies a physical clock realization, not a completed work-extraction protocol. The earlier FRW equation of state is retained with its different potential and signed coefficient. The changing-w term specifies when radius work alone is incomplete, and a general moving-FRW canonical theorem remains open. The companion preserves further conditional applications and their separate hypotheses.

Declaration of generative AI and AI-assisted technologies

During preparation and revision, the author used OpenAI ChatGPT and Codex to assist with literature searches, algebraic checks, and presentation. The author takes full responsibility for the manuscript.

Data availability

No observational data are used. Supporting files contain the LaTeX sources and symbolic calculations.

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