# Horizon Boundary Mechanics: Force, Pressure, and Work from the Einstein Boundary Potential

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**Abstract**

Recognizing acceleration length as a geometrically normalized work coordinate of the established Einstein boundary potential identifies a generalized boundary force and normal-scale pressure. Their action-supported work interpretation joins the area-conjugate surface stress in one macroscopic state law. The selected nonexpanding-null action response supplies the principal canonical justification; the stationary area–boost route gives a complementary interpretation with shared clock and area normalization. Schwarzschild work becomes $M\kappa\, dR = P_R\, dV$, with $P_R = 2P_L$. The separately assigned instantaneous equilibrium FRW screen has $P_R = P_L = \rho/3$ and potential equal to the Misner–Sharp horizon energy. Exact published scalar overlaps are identified alongside the added length-work construction. Temperature and entropy enter after the mechanics.

*Keywords:* horizon boundary mechanics, gravitational boundary response, acceleration length, null symplectic structure, FRW cosmology

## 1. Geometry and the boundary potential

A horizon cut has transverse area $A$ and a surface-gravity scale $\kappa$, with its physical time normalization specified. Express the same acceleration scale as a length:

$$L_s = \frac{c^2}{\kappa}, \qquad L_\kappa = |L_s|, \qquad \kappa \neq 0. \tag{1}$$

In Rindler geometry $L_\kappa$ is the normal observer–horizon proper distance on the instantaneous rest slice. The characteristic time $t_\kappa = c/|\kappa|$ gives $L_\kappa = ct_\kappa$ and, in stationary normal flow, one unit of boost rapidity. Neither a photon transit from the horizon nor a universal spherical radius is assumed.

For the Einstein boundary prescription used here, the energy-valued potential is [5, 8, 9]

$$\boxed{U_\partial(A, L_s) = \frac{A\kappa c^2}{8\pi G} = \frac{c^4}{8\pi G}\frac{A}{L_s}.} \tag{2}$$

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The Einstein boundary structure fixes the normalization (Sec. 3), not dimensional analysis. With $\kappa_E = 8\pi G/c^4$, $1/\kappa_E$ has energy-per-length units and $A/L_s$ supplies the length factor.

What displacement pairs with this potential's work? Its $L_s$ dependence identifies a horizon-scale force alongside the area response. Here $U_\partial = Mc^2/2$ for Schwarzschild and $U_\partial = E_{\mathrm{MS}}$ for the equilibrium-FRW screen; Table 1 separates their inputs and support.

## 2. One differential, common mechanical responses

At fixed $c$ and $G$, differentiating before imposing a particular horizon family's relation between $A$ and $L_s$ gives

$$\begin{aligned} dU_\partial &= \frac{c^4}{8\pi G L_s}\, dA - \frac{Ac^4}{8\pi G L_s^2}\, dL_s \\ &= \Pi_\partial\, dA - F_L\, dL_s. \end{aligned} \tag{3}$$

Writing $M_\partial = U_\partial/c^2$ for the potential's mass equivalent, the coefficients are

$$\Pi_\partial = \left(\frac{\partial U_\partial}{\partial A}\right)_{L_s} = \frac{U_\partial}{A} = \frac{\kappa c^2}{8\pi G}, \tag{4}$$

$$F_L = -\left(\frac{\partial U_\partial}{\partial L_s}\right)_A = \frac{U_\partial}{L_s} = M_\partial \kappa. \tag{5}$$

The area stress and its length-conjugate force share one potential. The normal-scale pressure is

$$\boxed{P_L = \frac{F_L}{A} = \frac{\Pi_\partial}{L_s} = \frac{c^4}{8\pi G L_s^2} = \frac{\kappa^2}{8\pi G}.} \tag{6}$$

Pressure is a response density, not a third canonical pair; $F_L = M_\partial \kappa$ is an identity, not an additional material equation of motion.

The mechanical energy identity and differential are

$$\begin{aligned} U_\partial &= \Pi_\partial A = F_L L_s = P_L A L_s, \\ dU_\partial &= \Pi_\partial\, dA - F_L\, dL_s. \end{aligned} \tag{7}$$

Since $U_\partial = A\Pi_\partial$, $A\, d\Pi_\partial = -F_L\, dL_s$ and $d\Pi_\partial = -P_L\, dL_s$. Thus pressure is the negative stress slope across boundary configurations, not a material-layer gradient. The scale laws

$$\frac{1}{L_s} = \kappa_E \Pi_\partial, \qquad \frac{1}{L_s^2} = \kappa_E P_L \tag{8}$$

are consequences of the common mechanics, not its starting hierarchy.

No enclosed volume is needed. If $\vartheta_V = A\, dL_s$,

$$F_L\, dL_s = P_L \vartheta_V, \qquad d\vartheta_V = dA \wedge dL_s. \tag{9}$$

Thus $\vartheta_V$ is not a global volume differential on the independent source space. The unchanged $P_L$ multiplies $dV$ only when $dV = A\, dL_s$. A sphere instead has $dV = A\, dR$ and the transformed pressure below; $AL_s$ is not generally its volume.

## 3. Canonical foundation of the response

The action variation gives interior field equations and a boundary one-form $\Theta$, linear in a field variation like $p\,\delta q$. Its field-space derivative is the canonical two-form. Boundary terms select its source and response; $\Theta$ is distinct from the energy $U_\partial$.

Let a null hypersurface $\mathcal{N}$ be foliated by spacelike two-dimensional cuts $S$, with metric $q_{AB}$ and area element $\epsilon_S = \sqrt{q}\, d^2x$. For a selected null generator $\xi^a$, define

$$\nabla_\xi \xi^a = \widehat{\kappa}\xi^a, \qquad \theta = \frac{1}{\sqrt{q}} \mathcal{L}_\xi \sqrt{q}. \tag{10}$$

Inaffinity $\widehat{\kappa}$ measures nonaffine parameterization; $\theta$ measures fractional area change. Both have inverse-length units, with $\kappa = c^2\widehat{\kappa}$. A null generator has no proper time or material proper acceleration.

Hopfmüller and Freidel identify the scalar canonical combination [8, 9]

$$\mu = \widehat{\kappa} + \frac{1}{2}\theta. \tag{11}$$

Use a fixed Gaussian-null patch $\mathcal{N} = \{r = 0\}$ with $\xi = \partial_v$, $g_{vr} = 1$, and current orientation $-dr$. Fix the chart, generator, normal data, cut identifications and interval. In units $c = 1$, define

$$B_0 = \frac{1}{8\pi G} \int_{\mathcal{N}} \epsilon_S (\widehat{\kappa} + \theta)\, dv.$$

Subtracting $B_0$ from the EH action gives the scalar area polarization $\Theta_A^{(0)} = -(8\pi G)^{-1} \int \mu\, \delta\epsilon_S\, dv$. For $B_\mu = (8\pi G)^{-1} \int \mu \epsilon_S\, dv$, select $I_{\text{rate}} = I_{\text{EH}} - B_0 + B_\mu$. Its scalar response is

$$\Theta_\mu^{(0)} = \Theta_A^{(0)} + \delta B_\mu = \frac{1}{8\pi G} \int_{\mathcal{N}} \epsilon_S\, \delta\mu\, dv. \tag{12}$$

This polarization exchange, $-p\delta q + \delta(pq) = q\delta p$, leaves the two-form unchanged. Covariant mixed conditions identify the same derivative-containing source [10]; Table 2 specifies the correspondence.

Restrict to the nonexpanding sector and variations tangent to it:

$$\theta = 0, \qquad \delta\theta = 0. \tag{13}$$

Control shear, matter and remaining boundary/corner responses. A sufficient local fixed-cut-metric variation has $\delta q_{AB} = \partial_v \delta q_{AB} = 0$. Restoring $\widehat{\kappa} = \kappa/c^2$ and the fixed clock map $dv = c\, d\tau$ gives

$$\Theta_\kappa^{(0)} = \frac{c^2}{8\pi G} \int d\tau \int_S \epsilon_S\, \delta\kappa. \tag{14}$$

Table 1: Claim-to-evidence map. Family restrictions follow the independent-coordinate derivatives.

| Claim | Inputs or prescription | Derivation and support |
|---|---|---|
| Force and pressure | Specified Einstein potential; independent $(A, L_s)$; fixed $c, G$. | Eqs. (2)–(9): conditional algebra. |
| Canonical length response | Selected rate action; fixed clock, cuts and interval; $\theta = \delta\theta = 0$; controlled other responses. | Eqs. (10)–(17); full Appendix I: restricted action derivation. |
| Geometric length | Uniformly accelerated Rindler observer, orthogonal rest slice, independent planar area. | Sec. 5.1; full Sec. V A: geometric normalization. |
| Schwarzschild radius work | Infinity-normalized clock; $L_\kappa = 2R_H$; $U_\partial = Mc^2/2$; $dV_H = A_H dR_H$. | Sec. 5.2, Eqs. (27)–(28): exact family identities. |
| FRW screen identities | Einstein–Misner–Sharp relation $E_A = \rho V_A$ and separate CK rate $c^2/R_A$. | Eqs. (30)–(37): assigned screen-state identities. |
| Thermal interpretation | Hawking–Unruh/KMS normalization and Einstein entropy with zero additive constant. | Eq. (22), Sec. 6: additional physical dictionary. |

The coordinate change follows directly from $L_s = c^2/\kappa$: $\delta\kappa = -c^2\delta L_s/L_s^2$. Substituting it into the gravitational response gives

$$\boxed{\begin{aligned}\Theta_L^{(0)} &= -\int d\tau \int_S \epsilon_S \, \frac{c^4}{8\pi G L_s^2}\, \delta L_s \\ &= -\int d\tau \int_S \epsilon_S P_L \, \delta L_s .\end{aligned}} \tag{15}$$

For a common length variation, $F_L = \int_S P_L \, dA$; a uniform finite cut has $F_L = AP_L$ and $\Theta_L^{(0)} = -\int d\tau F_L \delta L_s$. Thus $F_L$ is the coefficient of an allowed length-source variation in the selected action. This canonical role does not require $U_\partial$ to be the entire Hamiltonian. Pressure is its area density.

Equivalently, $\Theta_\kappa^{(0)}/\Delta\tau = A\delta\Pi_\partial$ for uniform stationary data. The exchange $-\Pi_\partial\delta A + \delta(A\Pi_\partial) = A\delta\Pi_\partial$ describes the same structure; neither selected one-form alone equals $dU_\partial$.

Source variations compare admitted nearby boundary-value problems, preserving nullness and tangent nonexpansion. Within each conservative problem the mixed source is fixed; the shared clock prescription still permits $\delta\kappa \neq 0$ between geometries.

Equivalently, the mixed on-shell action has local Hamilton–Jacobi response

$$F_L(\tau) = -\frac{\delta I_{\rm os}}{\delta L_s(\tau)}. \tag{16}$$

The time measure makes this a force; a constant-source derivative over a fixed interval is instead $-\Delta\tau F_L$. The action and counterterm prescription matter. Arbitrary source-dependent additions can shift the response, and arbitrary-source existence is not asserted.

In this scalar sector,

$$\frac{\mathbf{\Omega}^{(0)}}{\Delta\tau} = \delta A \wedge \delta\Pi_\partial = \delta L_s \wedge \delta F_L. \tag{17}$$

These represent one pair. The Legendre term selects rate polarization; $\kappa \mapsto c^2/\kappa$ changes its coordinate. Among lengths built only from $c$ and $\kappa$, dimensions give $Cc^2/\kappa$ and Rindler distance fixes $C = 1$. Other coordinates preserve work with transformed coefficients.

The net action shift is $-(A_f - A_i)/(16\pi G)$ $(c = 1)$. Tangent nonexpansion throughout the fixed interval gives $\delta A_f = \delta A_i$, controlling this endpoint term. Full-companion Appendix I derives the EH current from metric and variation jets and displays the remaining responses and counterexamples to unrestricted extensions.

### *3.1. Stationary area–boost interpretation*

A spacelike corner carries an area–boost pair [11, 12]. Choose its canonical orientation consistently with Eq. (15). The action-valued coefficient and the rate of the dimensionless normal boost angle are

$$J_\eta = \frac{Ac^3}{8\pi G}, \qquad \omega_s = \frac{\kappa}{c} = \frac{c}{L_s}. \tag{18}$$

Table 2: Closest canonical antecedents and the added interpretation. Source units and normalizations are matched first.

| Earlier source and equations | Established foundation adopted | Length-coordinate interpretation here |
|---|---|---|
| Parattu et al. [5], (111)–(113). | Static area-weighted rate response $A\delta\kappa/(8\pi G) = S\,\delta T$ ($c = 1$). | $L_s = c^2/\kappa$ gives $-F_L\delta L_s$ and $P_L = F_L/A$; Eqs. (6), (15). |
| Hopfmüller–Freidel [8], (1.1); [9], (1.3)–(1.4). | Area/scalar pair, $\mu = \widehat{\kappa} + \theta/2$, and the rate potential $\epsilon_{\mathcal{N}}\delta\mu$. | Tangent nonexpansion and the area measure give the selected length response, Eqs. (12)–(17). |
| Odak et al. [10], (3.26)–(3.28), (4.6), (4.8). | Scalar polarizations, mixed source $\theta + 2k = 2\mu$ ($k = \widehat{\kappa}$: null inaffinity), and a separate corner condition. | Match the scalar source; our selected action and endpoints are specified in full Appendix I, not identified with every mixed problem. |

The boost changes the time–normal frame, not spatial rotation. For the same stationary clock, charge times rate gives

$$\boxed{U_\partial = J_\eta\omega_s = \frac{J_\eta}{L_s/c}, \qquad U_\partial L_s = cJ_\eta.} \tag{19}$$

The action coefficient supplies $c^3$ and the rate $c/L_s$ supplies the remaining $c$, using the same normalized length and energy.

At fixed stationary duration $\Delta\tau$,

$$\eta = \frac{c\Delta\tau}{L_s}, \qquad \delta\eta = -\frac{c\Delta\tau}{L_s^2}\delta L_s. \tag{20}$$

The corner one-form therefore gives

$$\frac{J_\eta\delta\eta}{\Delta\tau} = -\frac{cJ_\eta}{L_s^2}\delta L_s = -F_L\delta L_s. \tag{21}$$

This is a complementary recovery with shared area normalization, clock and stationary boost relation. The selected corner one-form is not a complete Hamiltonian variation, nor an arbitrary cut's prescribed boost rate automatically its evolution rate.

### *3.2. Scope of the action result*

Stationary Rindler, Schwarzschild and consistently normalized de Sitter realize this construction. At fixed $\Lambda$, the cosmological term contains no metric derivatives and adds no derivative symplectic-potential term. A vanishing null expansion on each FRW cut does not make the worldtube nonexpanding or null: flat dust has $R_A = 3ct/2$, horizon speed $c/2$ relative to comoving flow, and a timelike tube.

On an expanding null boundary the source is $\mu$; rewriting $\delta\kappa$ retains expansion and other sectors. The FRW identities below instead use their explicit CK calibration, with a different canonical provenance.

## 4. Boundary potential versus mass first law

The mass first law is a Hamiltonian variation for a specified evolution, including the symplectic-potential subtraction and any generator corrections; it is not the full differential of $A\kappa$. Thus $d(TS) = T\,dS + S\,dT$ does not add $S\,dT$ to that law. Surface-Hamiltonian treatments already give the complementary terms [5, 4]; here the rate term acquires the length-force and pressure interpretation (Table 2). Thermally, the source exchange is $-T\delta S + \delta(TS) = S\,\delta T$.

Where the positive-branch thermal identification applies, differentiating $U_\partial = \omega_s J_\eta$ gives

$$\begin{aligned} \omega_s\,dJ_\eta &= \Pi_\partial\,dA = T\,dS, \\ J_\eta\,d\omega_s &= -F_L\,dL_\kappa = S\,dT. \end{aligned} \tag{22}$$

The scale leg has exterior derivative $(c^2/8\pi G)\delta A \wedge \delta\kappa$, whereas $dU_\partial$ is exact. Its integrability along a one-dimensional family does not make it an independent state energy.

For Schwarzschild, $E = Mc^2 = 2U_\partial$ and the constrained family gives

$$dE = T\,dS = 2F_L\,dL_\kappa, \qquad dU_\partial = F_L\,dL_\kappa. \tag{23}$$

Appending the boundary scale term as an independent addition to $dE = T\,dS$ would double-count that variation.

## 5. Stationary and cosmological realizations

Table 1 separates stationary realizations from assigned cosmological screens. Static de Sitter uses positive-magnitude $L = R_A$. On a spherical marginal horizon, $A = 4\pi R^2$ and the Misner–Sharp energy is $E_{\rm MS} = c^4 R/(2G)$ [15, 16]. The area potential therefore has the spherical form

$$\boxed{U_\partial = E_{\rm MS}\frac{R}{L_s}, \qquad U_\partial L_s = cJ_\eta = E_{\rm MS}R.}$$

For $\kappa > 0$, $U_\partial/E_{\rm MS} = R/L_\kappa$ is one in Cai–Kim FRW and one half in Schwarzschild. This realizes the area potential on a marginal horizon, not an arbitrary sphere; differentials retain $E_{\rm MS} \propto R$.

### 5.1. Rindler: the acceleration-length interpretation

For an observer with constant proper acceleration $a$ in Minkowski spacetime,

$$\kappa = a, \qquad L_\kappa = c^2/a, \qquad P_L = a^2/(8\pi G). \quad (24)$$

The normal rest-slice distance is $c^2/a$ and planar area $A = \int dy\, dz$ is independent. No source mass or spherical projection is required. Related energy constructions exist [13]; the present rate, length and potential share Eq. (1)'s normalization.

### 5.2. Schwarzschild: work of horizon-size change

With stationary Killing time normalized at infinity and areal radius $R_H$ defined by $A_H = 4\pi R_H^2$,

$$\kappa = \frac{c^2}{2R_H}, \quad L_\kappa = 2R_H, \quad F_L = \frac{c^4}{8G}, \quad U_\partial = \frac{Mc^2}{2}. \quad (25)$$

The Smarr relation becomes

$$\boxed{Mc^2 = TS + F_L L_\kappa, \qquad TS = F_L L_\kappa = \frac{Mc^2}{2}.} \quad (26)$$

The products coincide pointwise and represent different derivative directions of one potential, not independently stored energies. The same work expressed through the actual horizon size is

$$\boxed{\begin{aligned} F_R = 2F_L &= \frac{U_\partial}{R_H} = M\kappa, \\ P_R = \frac{F_R}{A_H} &= 2P_L = \frac{\kappa^2}{4\pi G}, \\ F_L\, dL_\kappa &= F_R\, dR_H = P_R\, dV_H. \end{aligned}} \quad (27)$$

Here $V_H = 4\pi R_H^3/3$ and $dV_H = A_H\, dR_H$. Since $dL_\kappa/L_\kappa = dR_H/R_H$, length work measures the same fractional horizon-size change, with no displacement of another surface at $2R_H$. The areal radius is a geometric measure of size, not a universal proper distance. The complete law is

$$\boxed{\begin{aligned} dU_\partial &= \Pi_\partial\, dA_H - M\kappa\, dR_H \\ &= T\, dS - P_R\, dV_H. \end{aligned}} \quad (28)$$

Area and volume co-vary along this family: $\Pi_\partial\, dA_H = 2P_R\, dV_H$, $dU_\partial = P_R\, dV_H$, and $U_\partial = 3P_R V_H$. The mass law remains Eq. (23). Transforming the displacement and its coefficient preserves the selected canonical response of Sec. 3; it adds no independent source variation.

### 5.3. Cosmological application: the FRW screen

For four-dimensional FRW with arbitrary spatial curvature,

$$R_A = \frac{c}{\sqrt{H^2 + kc^2/a^2}}, \qquad H = \dot{a}/a. \quad (29)$$

The geometric Misner–Sharp energy at a marginal sphere and its Einstein–FRW matter identification give [15, 22]

$$E_A = \frac{c^4 R_A}{2G} = \rho V_A, \qquad V_A = \frac{4\pi R_A^3}{3}. \quad (30)$$

Here $\rho$, $p_m$ are bulk energy density and pressure in the same units. Areal volume $V_A$ is not generally proper spatial volume for $k \neq 0$. The equality $E_A = \rho V_A$ is the Einstein matter–geometry input, not a result of defining pressure.

The equilibrium FRW (instantaneous Cai–Kim) prescription [14] sets

$$\kappa_{\rm CK} = \frac{c^2}{R_A}, \qquad L_{\rm CK} = R_A. \quad (31)$$

Then the force, screen pressure, and integrated identity are

$$F_L^{\mathrm{CK}} = \frac{c^4}{2G}, \quad P_{\mathrm{scr}} = P_L^{\mathrm{CK}} = \frac{c^4}{8\pi G R_A^2} = \frac{\rho}{3}, \quad (32)$$

$$E_A = \rho V_A = U_\partial = TS = F_L L_{\mathrm{CK}} = 3P_{\mathrm{scr}} V_A. \quad (33)$$

Here $L_{\mathrm{CK}} = R_A$ gives $F_R = F_L = M_A \kappa_{\mathrm{CK}}$ and $P_R = P_L = P_{\mathrm{scr}}$, with $M_A = E_A/c^2$: acceleration-length and radius work coincide. The thermal equality uses the CK temperature and Einstein entropy. The complete constrained differential is

$$dE_A = \Pi_{\mathrm{CK}}\, dA_A - P_{\mathrm{scr}}\, dV_A = T_{\mathrm{CK}}\, dS_A - P_{\mathrm{scr}}\, dV_A. \quad (34)$$

Along the spherical family, $dE_A/dV_A = P_{\mathrm{scr}}$. This total slope differs from the unconstrained derivative $-(\partial E/\partial V)_S$ and from cosmic material pressure.

Equation (34) is a screen-state law. Cai and Kim's matter-supply relation across a momentarily fixed horizon is $\delta Q_{\mathrm{matt}} = A_A(\rho + p_m) H R_A\, dt = T_{\mathrm{CK}} dS_A$ with orientation signs. One sorts change by boundary coordinate; the other sorts it by carrier. Exact mechanical state identities do not by themselves establish exact equilibrium thermality during arbitrary FRW evolution.

With $\kappa_E = 8\pi G/c^4$, the pressure representation is

$$\frac{1}{R_A^2} = \frac{H^2}{c^2} + \frac{k}{a^2} = \kappa_E P_{\mathrm{scr}}, \quad (35)$$

$${}^{(2)}\mathcal{R}_\perp \equiv \frac{2\ddot{a}}{ac^2} = -\kappa_E(P_{\mathrm{scr}} + p_m), \quad (36)$$

$$\dot{P}_{\mathrm{scr}} + H(3P_{\mathrm{scr}} + p_m) = 0. \quad (37)$$

$1/R_A^2$ is the horizon sphere's Gaussian curvature; ${}^{(2)}\mathcal{R}_\perp$ is the Ricci scalar of the normal time–radial metric, not the full spacetime Ricci scalar. Screen pressure sets the horizon scale, its sum with bulk pressure governs acceleration, and bulk enthalpy density $\rho + p_m = 3P_{\mathrm{scr}} + p_m$ governs its evolution. These give a screen-pressure representation of the same Friedmann dynamics.

The signed Hayward cross-focusing rate is different [16, 17]:

$$\frac{\kappa_H^s}{\kappa_{\mathrm{CK}}} = \frac{p_m - P_{\mathrm{scr}}}{4P_{\mathrm{scr}}} = \frac{3w-1}{4}, \qquad w = p_m/\rho. \quad (38)$$

In radiation FRW it vanishes while $P_{\mathrm{scr}}$ remains finite; in de Sitter their magnitudes agree. On a nonzero branch set $L_H^s = c^2/\kappa_H^s$ and define $F_L^H$ by Eq. (5): differentiate at fixed $A$, then evaluate at $(A_A, L_H^s)$. The quadratic response is $P_L^H = (P_{\mathrm{scr}} - p_m)^2/(16P_{\mathrm{scr}})$, and its scale work obeys

$$F_L^H\, dL_H^s = \frac{p_m - P_{\mathrm{scr}}}{4}\, dV_A - \frac{3}{4} E_A\, dw. \quad (39)$$

The volume coefficient at fixed $w$ is $P_R^H = (p_m - P_{\mathrm{scr}})/4$, with $P_L^H = (P_R^H)^2/P_{\mathrm{scr}}$. Along a varying-$w$ history the displayed $dw$ term remains. The pointwise reconstruction $p_m = P_{\mathrm{scr}} + 4P_R^H$ requires no constant-$w$ assumption; it uses the Einstein–FRW equations. $P_R^H$ is signed and distinct from Hayward work density.

*Exact published overlaps.* Kong's Eqs. (3.6)–(3.10) [18] give $P_{\mathrm{eff}} = -dE_A/dV_A = -P_{\mathrm{scr}}$ and $E_A = -3P_{\mathrm{eff}} V_A = 3P_{\mathrm{scr}} V_A$ in four-dimensional Einstein FRW. These match the scalar pressure magnitude and energy relation. Here the length response is defined on independent boundary sources before the spherical restriction, retaining both legs of $dU_\partial$.

In geometrized units $G = c = 1$, Ong's Eqs. (10)–(16) [21] give the shell energy $E = M/2 = TS$; the paragraph before Sec. V explicitly gives the spherical alternative $P = 1/(16\pi R_H^2) = P_R$. These are exact overlaps. Our radius work follows from $L_\kappa = 2R_H$ and the selected action, with spherical scaling $U_\partial = 2TS - 3P_R V_H$ and no shell-thickness or maximum-force premise. The companion's "Equation-level comparison with Kong and Ong" gives both derivations side by side.

## 6. Thermal translation and entropy density

A regular Hawking–Unruh/KMS state independently supplies the thermal normalization [7, 19, 20]. Writing $\Omega_\kappa = |\kappa|/c$, Euclidean regularity gives

$$c\Delta\tau_E = 2\pi L_\kappa, \quad k_B T = \frac{\hbar\Omega_\kappa}{2\pi}, \quad \frac{S}{k_B} = \frac{2\pi J_\eta}{\hbar}. \quad (40)$$

The last identity uses Einstein entropy with the standard zero additive constant. On the positive branch,

$$dU_\partial = T\, dS - F_L\, dL_\kappa, \qquad \frac{S}{k_B} = \frac{2\pi U_\partial L_\kappa}{\hbar c}. \quad (41)$$

Thus $U_\partial L_\kappa/c$ has the established meaning of boost charge before its conversion to dimensionless entropy. The area coefficient divided by temperature gives

$$\frac{S}{A} = \frac{P_L L_\kappa}{T} = \frac{k_B}{4\ell_P^2}, \qquad \ell_P^2 = \frac{G\hbar}{c^3}. \tag{42}$$

The quarter arises from the Einstein $1/(8\pi G)$ and thermal $1/(2\pi)$ normalizations.

## 7. Generality, scope, and conclusion

The contribution is the action-supported length-force and normal-pressure interpretation: Rindler geometry normalizes the length, the selected rate response supplies its work coefficient, and spherical restriction gives radius work. The potential, scalar canonical structure and published special-case coefficients are acknowledged inputs and overlaps (Table 2 and the comparisons above). The changing-$w$ term specifies when radius work alone is incomplete. The companion's moving-boundary and higher-curvature applications retain their own premises and derivations.

## Declaration of generative AI and AI-assisted technologies

During preparation and revision, the author used OpenAI ChatGPT and Codex to assist with literature searches, algebraic checks, and presentation. The author takes full responsibility for the manuscript.

## Data availability

No observational data are used. Supporting files contain the LaTeX sources and symbolic calculations.

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