# Horizon Boundary Mechanics: Force, Pressure, and Work from the Einstein Boundary Potential

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Acceleration length supplies a geometrically normalized work coordinate for the established Einstein horizon boundary potential. Writing $L_{\rm s} = c^2/\kappa$ and $U_\partial = A\kappa c^2/(8\pi G)$ gives $dU_\partial = \Pi_\partial\, dA - F_L\, dL_{\rm s}$, with $F_L = U_\partial/L_{\rm s} = (U_\partial/c^2)\kappa$ and $P_L = F_L/A = \kappa^2/(8\pi G)$. The contribution is the force and normal-pressure interpretation of the established rate response. A direct Einstein–Hilbert calculation in fixed Gaussian-null coordinates derives that response for a specified boundary action, clock, and tangent nonexpanding sector; it retains the endpoint-area term and keeps the conditions on the remaining response channels explicit. The stationary area–boost pair supplies a complementary interpretation. Rindler fixes the length normalization, while Schwarzschild converts the same work to areal radius, $F_L\, dL_{\rm s} = M\kappa\, dR$, with $L_{\rm s} = 2R$ and $U_\partial = Mc^2/2$. The separately calibrated equilibrium FRW screen has $L_{\rm s} = R_A$, $U_\partial = E_{\rm MS}$, and $P_{\rm scr} = \rho/3$. Together with the signed Hayward radial response it reconstructs the bulk pressure as $p_{\rm m} = P_{\rm scr} + 4P_R^H$ within Einstein–FRW dynamics. Thermal identification follows the mechanics. A claim-to-evidence map, equation-level source comparisons, and reproducible symbolic calculations distinguish the restricted action result, exact prescribed-screen identities, and conditional extensions. The dynamical charge–flux and stationary higher-curvature applications retain their own hypotheses; a general moving-FRW canonical theorem remains open.

## HOW TO READ THIS PAPER

The opening route identifies the length-conjugate force and normal-scale pressure through the full differential of the established Einstein boundary potential. The motivating question is whether horizon work can be paired with a geometric length, and with the same radius that controls area in the spherical realizations. The mechanical identity and first law precede temperature and entropy. Part II derives the force as the coefficient of an allowed length-source variation in the selected null-boundary action; this canonical role does not require the potential to equal the entire Hamiltonian. The stationary area–boost pair explains the same rate and normalization. Rindler, Schwarzschild, and stationary de Sitter realize this sector. Generic equilibrium FRW instead uses the Einstein–FRW constraint, marginal-horizon energy, and specified equilibrium rate. Detailed extensions retain their own source and geometric assumptions.

**Main results versus supporting material.** Parts I and II contain the mechanical construction, its canonical support, horizon realizations, and scope. Appendices A–C retain the sign conventions, FRW identities, and detailed timelike comparison. Appendix D collects exact source derivatives; Appendix E contains additional cosmological identities and their interpretation tests; Appendix F imports the standard inflationary spectrum as a separate application. Appendices G and H preserve interesting interpretations and conditional benchmarks without presenting them as independently derived physics. Appendix I gives the direct local Einstein–Hilbert calculation and explicit counterexamples that determine its domain. The evidence map in Part I identifies what supports each claim before the detailed derivations.

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## PART I. BOUNDARY MECHANICS AND PRINCIPAL REALIZATIONS

**Common mechanics and canonical scope.** The common mechanics below follows from the displayed Einstein boundary potential with a prescribed physical flow, clock, and sign convention. Its normal form is not restricted to one horizon geometry; each realization supplies its rate and displacement dictionary. The canonical statement in Part II additionally specifies a mixed rate-source polarization, boundary and corner terms, and variations tangent to a non-expanding null sector. Its force is the coefficient of the admitted length-source variation; the potential need not be the entire Hamiltonian. Varying the rate source means comparing nearby boundary problems; it is not a variation that keeps that source fixed. Rindler, Schwarzschild, and consistently normalized de Sitter realize the stationary construction. Equilibrium FRW is a separate constrained-screen realization with an instantaneous-equilibrium calibration. The thermal representation is introduced after the mechanical response.

### Area and acceleration length

A horizon cut has transverse area $A = \int_S \epsilon_S$ and a normal acceleration scale set by the specified surface gravity. On a nonzero branch,

$$L_{\rm s} = \frac{c^2}{\kappa}, \qquad L_\kappa = |L_{\rm s}| = \frac{c^2}{|\kappa|}. \tag{S1}$$

Surface gravity is the regular geometric rate; acceleration length is its reciprocal mechanical scale. The characteristic time $t_\kappa = c/|\kappa|$ gives the light-travel length $L_\kappa = ct_\kappa$; in stationary normal flow $t_\kappa$ is the time for one unit of boost rapidity, not an assumed photon transit from a horizon. Rindler makes $L_\kappa = c^2/a$ a normal proper distance on the instantaneous rest slice. In equilibrium FRW $L_\kappa = R_A$; in Schwarzschild $L_\kappa = 2R_H$. The spherical work below uses the actual areal radius explicitly.

The specified Einstein boundary potential is

$$\boxed{U_\partial(A, L_{\rm s}) = \frac{A\kappa c^2}{8\pi G} = \frac{c^4}{8\pi G}\frac{A}{L_{\rm s}} = \frac{A}{\kappa_{\rm E} L_{\rm s}}, \qquad \kappa_{\rm E} = \frac{8\pi G}{c^4}.} \tag{S2}$$

The normalization comes from the Einstein boundary structure, whose source response is established in Part II. The inverse coupling has units of energy per length; $A/L_{\rm s}$ gives the geometric factor with length units. This dimensional observation explains the units but does not replace the gravitational input. A rectangular Rindler patch has $A = \Delta y\, \Delta z$; only a round sphere supplies $A = 4\pi R^2$.

The established potential is the input. The contribution is to identify a new generalized force and pressure by expressing its rate dependence through the normalized work length $L_{\rm s}$. This pairs the area response with a geometric force–displacement response. Conventional mass first laws need not be full differentials of this boundary potential.

### One differential identifies the mechanical responses

With the gravitational constants fixed, and before imposing a physical family's relation between the two boundary coordinates,

$$\begin{aligned} dU_\partial &= \frac{c^4}{8\pi G L_{\rm s}}\, dA - \frac{Ac^4}{8\pi G L_{\rm s}^2}\, dL_{\rm s} \\ &= \frac{U_\partial}{A}\, dA - \frac{U_\partial}{L_{\rm s}}\, dL_{\rm s}. \end{aligned} \tag{S3}$$

Define $M_\partial = U_\partial/c^2$. The area and negative length derivatives give

$$\Pi_\partial = \left(\frac{\partial U_\partial}{\partial A}\right)_{L_{\rm s}} = \frac{U_\partial}{A} = \frac{c^4}{8\pi G L_{\rm s}} = \frac{c^2\kappa}{8\pi G}, \tag{S4}$$

$$F_L = -\left(\frac{\partial U_\partial}{\partial L_{\rm s}}\right)_A = \frac{U_\partial}{L_{\rm s}} = M_\partial \kappa = \frac{A\kappa^2}{8\pi G}. \tag{S5}$$

They are two conjugate responses of the same gravitational potential. The mass–acceleration equality is an exact identity for its mass equivalent, not an independent material equation of motion. Pressure is their length-response density:

$$\boxed{P_L = \frac{F_L}{A} = \frac{\Pi_\partial}{L_\mathrm{s}} = \frac{c^4}{8\pi G L_\mathrm{s}^2} = \frac{\kappa^2}{8\pi G}.} \tag{S6}$$

For uniform data $F_L = AP_L$; more generally $F_L = \int_S P_L\, dA$ for a common length-source variation. The two partial derivatives separate the area and scale contributions; neither coordinate must remain fixed along an actual horizon evolution.

The mechanical energy identity and complete first law are

$$\boxed{U_\partial = \Pi_\partial A = F_L L_\mathrm{s} = P_L A L_\mathrm{s},} \tag{S7}$$

$$\boxed{dU_\partial = \Pi_\partial\, dA - F_L\, dL_\mathrm{s}.} \tag{S8}$$

Surface stress pairs with changing area, and the new generalized force pairs with changing horizon scale. The product rule explains why that work is contained in the potential: since $U_\partial = A\Pi_\partial$, the second contribution gives

$$A\, d\Pi_\partial = -F_L\, dL_\mathrm{s}, \qquad d\Pi_\partial = -P_L\, dL_\mathrm{s}, \qquad P_L = -\frac{d\Pi_\partial}{dL_\mathrm{s}}. \tag{S9}$$

The associated scale laws can now be read as consequences:

$$\frac{1}{L_\mathrm{s}} = \kappa_\mathrm{E}\Pi_\partial, \qquad \frac{1}{L_\mathrm{s}^2} = \kappa_\mathrm{E} P_L. \tag{S10}$$

No separate constant-product derivation is required to establish the pressure slope. It is a boundary-configuration response, not a spatial pressure gradient through a material layer. Surface stress, force, and pressure share a foundation; pressure is not a third independent canonical pair.

No enclosed volume is needed for these definitions. For $\vartheta_V = A\, dL_\mathrm{s}$,

$$F_L\, dL_\mathrm{s} = P_L \vartheta_V, \qquad d\vartheta_V = dA \wedge dL_\mathrm{s}, \qquad d(AL_\mathrm{s}) = A\, dL_\mathrm{s} + L_\mathrm{s}\, dA. \tag{S11}$$

The unchanged $P_L$ multiplies $dV$ only where $dV = A\, dL_\mathrm{s}$. For spherical areal volume, $dV = A\, dR$, the same work instead uses the transformed pressure $P_R$ described below. The product $AL_\mathrm{s}$ is not generally that volume. On the negative branch $U_\partial$ and $L_\mathrm{s}$ are signed; thermal identities below use the stated positive magnitudes.

### Boost-charge interpretation of the same mechanics

The normal-rate potential can be read before any thermal identification. Let $J_\eta$ denote the Einstein charge conjugate to a dimensionless normal boost angle, and let $\omega_s$ be its signed rate:

$$J_\eta = \frac{Ac^3}{8\pi G}, \qquad \omega_s = \frac{\kappa}{c} = \frac{c}{L_\mathrm{s}}, \qquad \boxed{U_\partial = \omega_s J_\eta, \quad U_\partial L_\mathrm{s} = cJ_\eta.} \tag{S12}$$

The charge has units of action, not ordinary rotational angular momentum. Multiplication by the normal boost rate gives the energy-valued boundary potential. At fixed area its energy–length product is independent of the rate: $U_\partial L_\mathrm{s}/c = J_\eta$. With positive magnitudes the same relation is $|U_\partial| L_\kappa = cJ_\eta$. This is an explanatory consolidation of the established area–boost structure, not a new independent conserved energy. Its stationary canonical derivation is given in Part II.

The two contributions of the complete differential are

$$\boxed{dU_\partial = \omega_s\, dJ_\eta + J_\eta\, d\omega_s = \Pi_\partial\, dA - F_L\, dL_\mathrm{s}.} \tag{S13}$$

The first changes the area charge at a prescribed rate; the second changes the rate at prescribed charge. On the positive branch, only after Hawking–Unruh thermality is supplied, these become $T\, dS$ and $S\, dT$, respectively. Mechanical rate and charge should not be called thermal variables before that additional identification.

### Thermal translation and complementary scale laws

Where Hawking–Unruh/KMS thermality applies, it independently selects the same positive scale through $c\Delta\tau_E = 2\pi L_\kappa$. On the conventional positive thermodynamic branch, $U_\partial = TS$ and $\Pi_\partial\, dA = T\, dS$, so the same differential becomes

$$\boxed{dU_\partial = T\, dS - F_L\, dL_\kappa = T\, dS - P_L\, \vartheta_V.} \tag{S14}$$

Surface gravity sets two complementary laws of the same positive scale:

$$\boxed{k_B T = \frac{\hbar|\kappa|}{2\pi c} = \frac{\hbar c}{2\pi L_\kappa}, \qquad P_L = \frac{\kappa^2}{8\pi G} = \frac{c^4}{8\pi G L_\kappa^2}.} \tag{S15}$$

Hawking temperature is the semiclassical thermal law; $P_L$ is its classical mechanical counterpart, conjugate per unit area to a change of acceleration length. Combining the two normalizations gives

$$\boxed{\frac{k_B T}{P_L L_\kappa} = 4\ell_{\mathrm{P}}^2, \qquad \frac{S}{A} = \frac{P_L L_\kappa}{T} = \frac{k_B}{4\ell_{\mathrm{P}}^2}.} \tag{S16}$$

Pressure times acceleration length supplies the classical boundary energy per area; temperature converts it into the displayed entropy density. If entropy is reconstructed from $T\, dS = \Pi_\partial\, dA$, the integration gives $S = k_B A/(4\ell_{\mathrm{P}}^2) + S_0$; $S_0 = 0$ is the standard Einstein/Wald normalization used throughout.

### Realizations, clocks, and levels of support

The motivating work question is common, but its geometric and energy assignments differ. The following dictionary separates the potential from total energy and separates an action-source response from an exact constrained-screen identity. Canonical refers to the selected action and sources, not a requirement of time-dependent evolution.

| Realization | Prescribed rate and length | Potential and work |
|---|---|---|
| Rindler planar patch | Observer acceleration $a_{\mathrm{obs}} > 0$; $L = c^2/a_{\mathrm{obs}}$ is normal separation | $U = Ac^2 a_{\mathrm{obs}}/(8\pi G)$; independent area and normal length; no enclosed spherical mass or forced volume. |
| Schwarzschild | Infinity-normalized stationary clock; $L = 2R$ | $U = Mc^2/2$, not total ADM energy $Mc^2$; $F_L dL = M\kappa dR = P_R dV$, $P_R = 2P_L$. |
| Stationary de Sitter | Specified static-patch clock; $L = R_A$ for the positive magnitude | $U^+ = E_{\mathrm{MS}}$; signed orientations must match. Neighboring radii generally change the cosmological source. |
| Evolving FRW equilibrium screen | Instantaneous positive assignment $\kappa_{\mathrm{eq}} = c^2/R_A$, $L = R_A$ | $U_{\mathrm{eq}} = E_{\mathrm{MS}}$; $P_R = P_L = P_{\mathrm{scr}} = \rho/3$. Exact assigned screen law, distinct from crossing heat. |
| Evolving FRW Hayward prescription | Signed normal rate $\kappa_H^{\mathrm{s}}$; $L_H^{\mathrm{s}} = c^2/\kappa_H^{\mathrm{s}}$ only on a nonzero branch | $U_H^{\mathrm{s}} = \chi E_{\mathrm{MS}}$; $P_R^H = (p_{\mathrm{m}} - P_{\mathrm{scr}})/4$; changing $w$ retains $-3E_{\mathrm{MS}} dw/4$. $P_L^H$ is different. |

*Claim-to-evidence map.* An algebraic identity verifies a consequence of its inputs; a local action calculation additionally identifies a source and its response. The following map identifies the assumptions and derivations for each result. Appendix I gives the local action calculation; the stated boundary-data and physical assumptions remain part of each application.

| Claim and derivation | Inputs retained | Support and remaining scope |
|---|---|---|
| Mechanical normal form; Eqs. (S2)–(S6) | Einstein potential, fixed $c, G$, prescribed clock, independent source coordinates $A, L_{\rm s}$ | Exact conditional derivatives. The mass–acceleration equality is the same identity, not another test. |
| Selected null action response; Sec. II, Appendix I | Fixed Gaussian-null domain; $I_{\rm EH} - B_0 + B_\mu$; $\theta = \delta\theta = 0$; other responses controlled | Direct local EH current, endpoint-area proof, and explicit counterexamples. Other boundary/corner responses must be controlled; existence of any chosen on-shell source family remains an assumption. |
| Stationary length and radius work; Sec. V | Specified Rindler observer, infinity-normalized Schwarzschild clock, or de Sitter static clock | Geometric specialization and algebra. The stationary corner route shares the normalization and is complementary. |
| Membrane gradient and normal balance; Sec. IV | Established membrane stress or screen projection; matched clock, normal and displacement | Exact pullback or rewriting of the imported force-balance equation; no new field equation. |
| Equilibrium FRW and signed Hayward reconstruction; Sec. VI, Eqs. (160)–(161) | Einstein–FRW equations, Misner–Sharp energy, specified equilibrium or Hayward rate | Exact on-shell identities. The moving timelike comparison in Appendix C uses a different source and demonstrates a mismatch. |
| Finite dynamics and Wald response; Secs. VII–VIII | APS preferred projection/generator; or stationary Wald channel and specified Lovelock sector | Imported theorems plus checked length-coordinate consequences. Their hypotheses are not supplied by the local null calculation. |
| Supporting applications; Appendices D–F | Stated fixed variables, Einstein cosmology, and, for inflation, the standard perturbation spectrum | Conditional derivatives and substitutions, with separate extension checks. No independent primordial-spectrum prediction. |
| Interpretations and benchmarks; Appendices G–H | Additional energy identifications, semiclassical extrapolations or a proposed minimum scale | Exact algebra where displayed; the additional physical identifications and cutoffs remain assumptions. |

**Stationary and cosmological realizations**

*Equilibrium FRW (Cai–Kim prescription).* For an FRW apparent horizon in the instantaneous Cai–Kim prescription,

$$L_\kappa = R_A, \qquad U_\partial = E_A, \qquad P_L = \frac{\rho}{3}, \qquad \boxed{dE_A = T\,dS_A - P_L\,dV_A.} \tag{S17}$$

Here acceleration length is the areal screen radius, so $F_R = F_L$, $P_R = P_L$ and $F_L\,dL_\kappa = F_R\,dR_A = P_R\,dV_A$, where $V_A = 4\pi R_A^3/3$ is the areal volume. For nonzero spatial curvature it is not the proper volume of the cosmic-time spatial ball. With the Einstein coupling $\kappa_{\rm E} \equiv 8\pi G/c^4$, the Friedmann constraint and acceleration equation take the compact pressure form

$$\boxed{\frac{1}{R_A^2} = \kappa_{\rm E} P_L, \qquad \frac{2\ddot{a}}{ac^2} = -\kappa_{\rm E}(P_L + p_{\rm m}).} \tag{S18}$$

Here $P_L$ fixes the transverse horizon-curvature scale, while $P_L + p_{\rm m}$ fixes the normal expansion curvature. Full Kodama–Hayward cross-focusing instead depends on the pressure contrast,

$$\boxed{\kappa_H^{\rm s} = -\frac{c^2 R_A}{4}\kappa_{\rm E}(P_L - p_{\rm m}), \qquad P_L^H = \frac{(P_L - p_{\rm m})^2}{16 P_L}.} \tag{S19}$$

Equivalently, Faraoni's established trace dependence can be packaged as

$$\Sigma_{\rm tr} \equiv \frac{(\rho - 3p_{\rm m})V_A}{c^2 A_A} = \frac{P_L - p_{\rm m}}{\kappa_{\rm CK}}, \qquad \boxed{\kappa_H^{\rm s} = -2\pi G \Sigma_{\rm tr}.} \tag{S20}$$

Here $\Sigma_{\rm tr}$ is the signed bulk stress–energy trace, converted to a mass equivalent and distributed over the horizon area. It is not assumed to be a membrane energy density or a literal shell source. The detailed status and operational tests are treated in Sec. VI.

*Schwarzschild: work of horizon-size change.* For a nonrotating black hole with Killing time normalized at infinity,

$$L_\kappa = 2R_H, \qquad U_\partial = TS = F_L L_\kappa = \frac{Mc^2}{2}, \qquad F_L = \frac{c^4}{8G}. \tag{S21}$$

The Smarr relation becomes

$$\boxed{Mc^2 = TS + F_L L_\kappa, \qquad TS = F_L L_\kappa = \frac{Mc^2}{2}.} \tag{S22}$$

This is an exact Euler–Smarr balance. Both products equal the parent potential; the area and scale derivatives provide the distinct mechanical responses. In the actual areal radius, the same work is

$$\boxed{\begin{aligned} F_L\, dL_\kappa = F_R\, dR_H = M\kappa\, dR_H = P_R\, dV_H, \\ F_R = 2F_L = M\kappa = \frac{c^4}{4G}, \qquad P_R = \frac{F_R}{A_H} = 2P_L = \frac{\kappa^2}{4\pi G}. \end{aligned}} \tag{S23}$$

Here $A_H = 4\pi R_H^2$ and $V_H = 4\pi R_H^3/3$. The equality $dL_\kappa/L_\kappa = dR_H/R_H$ describes the same fractional horizon-size change; there is no displacement of another surface at $2R_H$. The areal radius measures geometric size, not a universal proper distance. Thus $dU_\partial = \Pi_\partial\, dA_H - M\kappa\, dR_H = \Pi_\partial\, dA_H - P_R\, dV_H$, with area and volume linked by the same $R_H$. Along this family,

$$d(TS) = d(F_L L_\kappa) = \frac{1}{2}\, d(Mc^2), \qquad Mc^2 - TS = F_L L_\kappa. \tag{S24}$$

The first equality follows from the common parent potential; the second identifies the scale product with the formal on-shell Helmholtz free energy. The different differential channels are $\Pi_\partial\, dA$ and $-F_L\, dL_\kappa$, not independently variable thermal and mechanical energies.

*Rindler.* For a uniformly accelerated observer in flat spacetime,

$$\kappa = a, \qquad L_\kappa = \frac{c^2}{a}, \qquad P_L = \frac{a^2}{8\pi G}, \qquad U_\partial = \frac{Aac^2}{8\pi G}. \tag{S25}$$

The acceleration length is the normal observer–horizon proper distance on the instantaneous rest slice, the accelerated worldline's curvature radius, and its Euclidean thermal-orbit radius. The planar patch area is independent of this distance; no spherical horizon is assumed.

### Spherical energy–length interpretation

On a spherical marginal horizon, the geometric Misner–Sharp value is $E_{\rm MS} = c^4 R/(2G)$. Substituting $A = 4\pi R^2$ into the same potential gives

$$\boxed{U_\partial = E_{\rm MS}\frac{R}{L_{\rm s}}, \qquad U_\partial L_{\rm s} = cJ_\eta = E_{\rm MS} R.} \tag{S26}$$

For $\kappa > 0$, the energy ratio is $U_\partial/E_{\rm MS} = R/L_\kappa$: it equals one for Cai–Kim FRW and one half for infinity-normalized Schwarzschild. This spherical interpretation of the area–boost product is not a new independent charge law. $E_{\rm MS} \propto R$ here, so it must vary with $R$ in a differential. No enclosed Misner–Sharp energy or spherical projection is required for the general Rindler patch potential.

### Radius work and the Smarr scaling identity

For a spherical cut, introduce

$$A = 4\pi R^2, \qquad M_R \equiv \frac{c^2 R}{2G}, \qquad \alpha_R \equiv \frac{L_{\rm s}}{R} = \frac{c^2}{\kappa R}, \qquad F_R \equiv \frac{U_\partial}{R} = M_R \kappa. \tag{S27}$$

Here $M_R$ equals the Misner–Sharp mass at a spherical marginal horizon; it is not a universal total mass for arbitrary screens. On either nonzero signed branch the exact change of variables gives

$$\boxed{F_L\, dL_{\rm s} = F_R\, dR + U_\partial \frac{d\alpha_R}{\alpha_R}, \qquad dU_\partial = F_R\, dR - U_\partial \frac{d\alpha_R}{\alpha_R}.} \tag{S28}$$

The area leg is $\Pi_\partial\, dA = 2F_R\, dR$. The product identities below hold throughout this source chart. At fixed $\alpha_R$, $F_R = c^4/(2G\alpha_R)$ is constant and the potential is linear in $R$:

$$\boxed{U_\partial = F_R R = 3P_R V, \qquad P_R \equiv F_R/A = \alpha_R P_L, \qquad R\left(\frac{\partial U_\partial}{\partial R}\right)_{\alpha_R} = U_\partial.} \tag{S29}$$

The areal volume is $V = 4\pi R^3/3$. Cai–Kim FRW has $\alpha_R = 1$, $M_R = M_A$, and $U_\partial = E_A$; Schwarzschild has $\alpha_R = 2$, $M_R = M$, and $Mc^2 = 2U_\partial = 2F_R R$. These are constrained scaling relations, not additional independent terms in the mass first law. The radius form makes Smarr homogeneity explicit. The same homogeneity is already present in the acceleration-length potential as $2A\,\partial_A U_\partial + L_s\,\partial_{L_s} U_\partial = U_\partial$. Sections II D and V F give the canonical normalization and the limits of the mass–radius interpretation, including rotating and charged examples.

The complete radius formulation is as general as the acceleration-length formulation within this spherical scalar sector. Only its truncation to $F_R dR$ requires a constant scale ratio. Their common foundation is the area–normal-rate pair: $A$ measures transverse size, whereas $L_s$ represents the selected normal rate.

### Boundary inverse-energy or Compton-equivalent consequence

Restrict this subsection to the positive-$\kappa$ branch. Any specified positive energy scale $E$ defines the quantum inverse-energy length $\hbar c/E$. Applying that construction to the pressure-carrying boundary potential gives

$$\bar{\lambda}_\partial \equiv \frac{\hbar c}{U_\partial}. \tag{S30}$$

It has the mathematical form of a reduced Compton wavelength, but $U_\partial/c^2$ is generally a boundary mass equivalent rather than particle rest mass. The Euclidean thermal period in light-travel units is $C_E \equiv c\Delta\tau_E = 2\pi L_\kappa$. Since $U_\partial = TS$,

$$\boxed{\frac{S}{k_B} = \frac{C_E}{\bar{\lambda}_\partial} = \frac{2\pi L_\kappa}{\bar{\lambda}_\partial}, \qquad \frac{\bar{\lambda}_\partial}{L_\kappa} = \frac{8\pi\ell_{\mathrm{P}}^2}{A}.} \tag{S31}$$

Horizon entropy therefore measures the separation between the macroscopic thermal scale of the selected normal flow and the quantum inverse-energy scale of its boundary potential. This is an exact macroscopic relation within the Einstein–Hawking framework, not a claim that the horizon is a particle or is composed of literal wavelength-sized cells. Equivalently,

$$S = \frac{2\pi k_B U_\partial L_\kappa}{\hbar c}, \tag{S32}$$

which reproduces the familiar energy–radius product in Schwarzschild and Cai–Kim FRW and suggests $U_\partial L_\kappa$ as the more general normal-flow boundary product.

## PART II. CANONICAL FOUNDATIONS, SCOPE, AND EXTENSIONS

**Purpose of Part II.** The compact derivation in Part I is self-contained once its inherited horizon-mechanics inputs are accepted. Part II identifies the selected source and response in Einstein boundary theory, connects the stationary corner description, and determines where the compact normal form and its further consequences apply.

*What this part establishes.* The equations above are algebraically complete once their stated horizon-mechanics inputs are accepted. Part II supplies the restricted action derivation connecting the source response to general relativity. It derives the permitted $L_{\mathrm{s}}$ source variation from the mixed null symplectic potential (Sec. II and the direct calculation in Appendix I), complementarily recovers $F_L$ from the area–boost corner action, and establishes the inherited clock and gauge meaning, signed branches and $\kappa = 0$ limit, membrane pullback and normal force balance, Cai–Kim/Hayward distinction, off-family integrability limit, finite dynamical support, and higher-curvature extensions. At $\kappa = 0$ the reciprocal chart moves to infinity and the regular parent description must be written in $\kappa$.

### FROM ONE ACTION TO SEVERAL BOUNDARY DESCRIPTIONS

Einstein gravity on a finite region has a bulk action, boundary terms, and corner terms. Its variation separates field equations from boundary responses, schematically

$$\delta I = \int_M E \cdot \delta\Phi + \Theta_{\partial M} + \delta I_{\mathrm{boundary}} + \delta I_{\mathrm{corner}}.$$

Here $\Phi$ denotes metric and matter fields, $E = 0$ denotes their equations, and $\Theta$ is a field-space one-form, not an energy function. Evaluating on solutions gives a Hamilton–Jacobi functional of the admitted boundary data. Changing its polarization exchanges which member of a canonical pair is prescribed. Changing its clock or reference terms can change the physical response, even when the bulk equations are unchanged.

The null scalar potential and stationary area–boost corner give two routes to the same coefficient in their common sector. The surface Hamiltonian supplies the area/rate differential interpretation. The Brown–York timelike stress, spherical Kodama–Noether potential, and matter-flux laws are related descriptions with different geometric or variational inputs; they are not automatically one identical Hamiltonian. A complete Hamiltonian variation can contain symplectic, generator, transport, and flux corrections to the Noether surface term [11, 12, 21]. The analysis below specifies these distinctions rather than introducing several unrelated force laws.

### I. INTRODUCTION

Part I introduced the area and reciprocal acceleration scale, wrote the Einstein boundary potential in those variables, and obtained its mechanical responses together. This part asks how the same length-work coefficient appears in the covariant gravitational boundary response, which conditions it requires, and how each horizon realization supplies its rate.

Horizon mechanics naturally singles out area and surface gravity. Horizon thermodynamics repackages the same pair as entropy and temperature. What is less often made explicit is the mechanical content of varying the normal rate itself. In a stationary first law the horizon generator and its clock normalization are normally held fixed, so the area variation is visible while the complementary $A\,\delta\kappa$ direction is absent or absorbed by the definition of the charge. This is appropriate for that ensemble, but it leaves open a different boundary-value question: if the prescribed normal rate is allowed to vary as a source, what is its response?

The selected clock, generator normalization, relational cut, and equilibrium or non-expanding conditions are not special restrictions invented for $P_L$. They are inherited from the same horizon-mechanics sector in which the standard first law, membrane surface stress, and Hawking temperature define the same $\kappa$. The added construction is to open the reciprocal normal-rate source direction and take the full differential of the Einstein boundary potential $U_\partial(A, L_{\mathrm{s}})$, rather than only the Hamiltonian mass variation for a prescribed evolution. The identification $U_\partial = TS$ belongs to the later thermal translation.

The reciprocal of the prescribed surface-gravity scale is

$$L_{\mathrm{s}} = \frac{c^2}{\kappa} = \frac{1}{\kappa_{\mathrm{E}}\Pi_\partial} \tag{1}$$

and turns that question into a mechanical one. Surface gravity is the regular geometric rate; acceleration length is its reciprocal mechanical scale. On each nonzero branch they contain the same information, but $\kappa$ remains the regular

parent coordinate at a zero crossing while $L_{\rm s}$ moves to infinity. The latter is the light-travel length associated with the inverse surface-gravity timescale. In Rindler spacetime it is exactly the proper distance from a uniformly accelerated observer to that observer's horizon. For a stationary black hole it is an inverse redshift-normalized surface-gravity scale. In spherical dynamics its Hayward analogue is an inverse normal cross-focusing scale.

On stationary horizons admitting a regular Hawking–Unruh/KMS state, the thermal meaning of this reciprocal scale is exact rather than a dimensional resemblance. Analytic continuation of the normal time turns Lorentzian boost flow into rotation in the Euclidean normal two-plane. The horizon cut is the fixed-point set of that rotation, smoothness fixes a $2\pi$ angular period, and $c\Delta\tau_E = 2\pi L_\kappa$. Equivalently, the first nonzero bosonic Matsubara frequency is $\omega_1^{\rm E} = |\kappa|/c$, so $L_\kappa = c/\omega_1^{\rm E}$. The Euclidean/KMS periodicity and Matsubara spectrum are established results; the structural use made here is to identify that same reduced thermal length as the mechanical boundary source whose Einstein-action response is pressure [7, 33–36].

The mechanical responses already found from the boundary potential obey:

$$d\Pi_\partial = -P_L\, dL_{\rm s}, \qquad P_L = -\frac{d\Pi_\partial}{dL_{\rm s}} = \frac{\Pi_\partial}{L_{\rm s}} = \frac{c^4}{8\pi G L_{\rm s}^2} = \frac{\kappa^2}{8\pi G}. \tag{2}$$

The common Einstein boundary potential and its normalization are:

$$U_\partial(A, L_{\rm s}) = \Pi_\partial A = \frac{A}{\kappa_{\rm E} L_{\rm s}}, \qquad \mathcal{T}_G \equiv \frac{1}{\kappa_{\rm E}} = \frac{c^4}{8\pi G}. \tag{3}$$

For uniform data, $F_L = AP_L = -(\partial U_\partial/\partial L_{\rm s})_A$: area integration and the potential derivative give the same generalized force. At the reduced level, Eqs. (1)–(3) form a self-contained conditional theorem: once the established parent potential and acceleration-length coordinate are accepted in the inherited horizon-mechanics sector, the force, pressure, work form, and principal sectoral corollaries follow. The purpose of the full analysis is to identify the allowed source variation in gravitational boundary theory, reproduce its coefficient through the complementary stationary corner action, and determine the clock, gauge, sign, integrability, and dynamical limits. Appendix I makes the restricted Einstein–Hilbert calculation directly reproducible.

The generic scalar source on a null boundary is not surface gravity alone but

$$\mu = \widehat{\kappa} + \frac{1}{2}\theta, \tag{4}$$

where $\widehat{\kappa} = \kappa/c^2$ is the inverse-length inaffinity and $\theta$ is the null expansion. The clean $\kappa \leftrightarrow L_{\rm s}$ source chart therefore applies canonically on a clock-anchored non-expanding sector, not on every moving horizon. Other sectors can support the same algebra only after an independently established prescription for their normal rate and parent $A\kappa$ potential has been supplied.

The paper develops four additional consequences of the source response. First, the signed and magnitude relations

$$\Pi_\partial = P_L L_{\rm s}, \qquad |\Pi_\partial| = P_L L_\kappa, \qquad \Pi_\partial \equiv \frac{c^2\kappa}{8\pi G}, \tag{5}$$

link the new normal pressure to the established two-dimensional membrane pressure. The link is stronger than a comparison of units: for a nonuniform signed source on a cut, $D_A\Pi_\partial = -P_L D_A L_{\rm s}$; equivalently, on a fixed-sign branch, $D_A|\Pi_\partial| = -P_L D_A L_\kappa$. This is the coordinate-free pullback of the source-space response one-form to the physical cut and turns the usual membrane pressure gradient into a spatial manifestation of the normal scale response.

Second, the same substitution converts the gravitational-screen Young–Laplace equation into a normal force balance, using the screen normal rate and matching clock specified in Sec. IV. With the orientation in which the surface tension is $\gamma = -\Pi_\partial$, the exact signed equation and its positive-$\kappa$ form are

$$\Delta p - \Pi_\partial H = \Delta p - P_L L_{\rm s} H = 0, \qquad \Delta p - P_L L_\kappa H = 0 \quad (\kappa > 0). \tag{6}$$

Here $H$ is the mean curvature and $\Delta p$ is the appropriate outside-minus-inside normal pressure. Equation (6) is not a new Einstein equation; it is an acceleration-length rewriting of an established normal projection of Einstein's equation. It does, however, answer directly the question of whether $P_L$ can enter a genuine force-balance law.

Third, $L_\kappa$ is a mechanical–thermal bridge scale:

$$L_\kappa = \frac{c^2}{|\kappa|} = \frac{\hbar c}{2\pi k_B T}. \tag{7}$$

Classical boundary mechanics assigns it the inverse-square law $P_L(L_\kappa) = c^4/(8\pi G L_\kappa^2)$, while quantum field theory assigns it the inverse-linear law $k_B T(L_\kappa) = \hbar c/(2\pi L_\kappa)$. Their ratio fixes the Planck-area entropy density.

Fourth, Schwarzschild displays an exact pointwise thermal–mechanical Euler–Smarr balance:

$$Mc^2 = TS + F_L L_\kappa, \qquad TS = F_L L_\kappa = \frac{Mc^2}{2}. \tag{8}$$

The two products are equal representations of the same potential even on the extended source space; its area and scale differential legs are distinct. Along the Schwarzschild family the products co-vary exactly, $d(TS) = d(F_L L_\kappa) = \frac{1}{2} d(Mc^2)$: adding mass increases both halves in lockstep because both products equal the same parent potential. The exact products and their derivatives are developed below. Optional kinetic/potential and free-energy comparisons are separated in Appendix G.

The contribution claimed here is a new generalized boundary force $F_L = U_\partial / L_\mathrm{s}$ and its normal-scale pressure $P_L = \kappa^2/(8\pi G)$, identified by recognizing acceleration length as the geometrically normalized work coordinate of the established Einstein boundary potential. The mixed-boundary source response supplies the action-level support; the Rindler and spherical realizations supply the geometric meaning. The new response quantities are distinguished from the known potential, linear membrane stress, and complementary rate variation. Their identification is a result within Einstein gravity, not a new fundamental interaction. The construction also organizes force-balance, clock, integrability, and sector-dependent extensions.

The remainder of Part II establishes the source, gauge, canonical, mechanical, and thermal structure in Sections II–IV. Section V treats stationary realizations and the thermal–mechanical Euler identities. Section VI develops the FRW screen law and Hayward focusing response. Section VII treats integrability and finite dynamical balance. Section VIII gives the Wald-entropy extension and Section IX states the contribution and its limits. Appendix C records a separate timelike boundary comparison, restricted FRW surface charges, and the unresolved problem of matching the full moving-screen response to a horizon rate.

## II. NON-EXPANDING NULL-BOUNDARY SOURCE-RESPONSE PROPOSITION

### A. Established null scalar pair

Let $\mathcal{N}$ be a null hypersurface foliated by spacelike cuts $S$. Let $\xi^a$ be a chosen null evolution vector tangent to $\mathcal{N}$, and let $q_{AB}$ and $\epsilon_S = \sqrt{q}\, d^2x$ be the induced metric and area form on a cut. Define the inaffinity and expansion by

$$\nabla_\xi \xi^a = \widehat{\kappa}_\xi \xi^a, \qquad \theta_\xi = \frac{1}{\sqrt{q}} \mathcal{L}_\xi \sqrt{q}. \tag{9}$$

The cut is a spacelike two-dimensional cross-section, whereas $\mathcal{N}$ is its three-dimensional null history. A null normal is also tangent to $\mathcal{N}$. Inaffinity measures failure of the selected generator parameter to be affine: its tangent differentiates into a multiple of itself. A null generator has no proper time, so this is not the proper acceleration of a material observer. The expansion is the fractional change of the area element along that generator. In the length-parameter convention both $\widehat{\kappa}_\xi$ and $\theta_\xi$ have inverse-length units. The shear instead describes area-preserving shape distortion. The scalar, vector, and trace-free tensor labels classify cut geometry; they do not posit extra matter species.

For the canonical formulas in this subsection set $c = 1$, take the null parameter $v$ to have units of length, and retain $8\pi G$ explicitly. Hopfmüller and Freidel derived the null symplectic potential without initially fixing the diffeomorphism gauge. In four dimensions its bulk part has the schematic spin decomposition [11]

$$\Theta^{\mathrm{bulk}}_{\mathcal{N}} = \frac{1}{8\pi G} \int_{\mathcal{N}} \left[ \frac{1}{2} \widetilde{\sigma}^{AB} \delta \widetilde{q}_{AB} - \overline{\eta}_A \delta \xi^A - \left( \widehat{\kappa}_\xi + \frac{1}{2} \theta_\xi \right) \delta \ln \sqrt{q} \right] \epsilon_{\mathcal{N}}. \tag{10}$$

The first two terms are the spin-two and spin-one sectors. The spin-zero momentum is

$$\boxed{\mu_\xi = \widehat{\kappa}_\xi + \frac{1}{2} \theta_\xi.} \tag{11}$$

Writing $\epsilon_{\mathcal{N}} = dv \wedge \epsilon_S$, the scalar area polarization is

$$\Theta^{(0)}_A = -\frac{1}{8\pi G} \int_{\mathcal{N}} \mu_\xi \, \delta \epsilon_S \, dv. \tag{12}$$

Adding the exact variation of

$$B_\mu = \frac{1}{8\pi G} \int_{\mathcal{N}} \mu_\xi \epsilon_S \, dv \tag{13}$$

changes the scalar polarization without changing the symplectic two-form:

$$\Theta^{(0)}_{\mu} = \Theta^{(0)}_{A} + \delta B_{\mu} = \frac{1}{8\pi G}\int_{\mathcal{N}} \epsilon_S\, \delta\mu_\xi\, dv. \tag{14}$$

The elementary operation is $-p\,\delta q + \delta(pq) = q\,\delta p$. The first form displays area as the source; the second displays its mixed rate. Because $\delta^2 B_\mu = 0$, this boundary Legendre transform leaves the scalar symplectic two-form unchanged. The superscript (0) labels the scalar sector, not perturbative order. The one-half in $\mu_\xi$ is the four-dimensional specialization of the null geometric decomposition, not a coefficient chosen to reproduce the pressure. Hopfmüller and Freidel interpret $\mu$ as a gravitational boundary pressure and argue that fixing it is a natural clock boundary condition for null translations [12]. This established pressure is linear in $\mu$ and is not the quadratic force-per-area response introduced below.

Odak, Rignon-Bret, and Speziale subsequently obtained the covariant conformal/York polarization for general null variations [13]. In their conventions the spin-zero source is $\theta + 2\widehat{\kappa} = 2\mu$. Their conservative mixed conditions fix that quantity together with appropriate spin-two, generator, and corner data. The scalar condition is Robin-type because it contains metric derivatives. The use of $\mu$ as a mixed-boundary source is established. For clarity, the displayed polarization is selected by the boundary Legendre term $B_\mu$, with orientation, cut identifications, generator, and remaining boundary and corner data prescribed. Its variation is evaluated on an extended space of sources. Conservative boundary conditions impose $\delta\mu = 0$ within each fixed-source problem; the response below is obtained by comparing nearby such problems. An additional source-dependent boundary functional can shift the response. No invariance under arbitrary counterterm changes is claimed.

*a. Equation-level source correspondence.* The following correspondence identifies the inherited null structure. Appendix I separately derives the local fixed-generator result directly from the EH current. For the matched generator, orientation and clock,

$$\widehat{\kappa} = \kappa/c^2, \qquad \mu = \widehat{\kappa} + \theta/2 \quad (D=4), \qquad \theta + 2k = 2\mu. \tag{15}$$

Here $k$ is the geometrical inaffinity of Ref. [13], not FRW spatial curvature or a different observer's clock. Its $\delta(\theta+2k)/(16\pi G)$ scalar density becomes $\delta\mu/(8\pi G)$ only after these conventions are matched. The null volume form retains the boundary-time measure, not cut area alone.

| Primary source/version | Checked correspondence | Restrictions retained |
|---|---|---|
| Hopfmüller–Freidel, 1611.03096v2 | Eqs. (4.27), (5.3)–(5.5): scalar/shear decomposition; $\widehat{\kappa} + (D-3)\theta/(D-2)$; $8\pi G = c = 1$ | Fixed foliation/time measure for the scalar reduction; source $L^a$ denotes a generator, not acceleration length. |
| Hopfmüller–Freidel, 1802.06135v1 | Eqs. (1.3)–(1.4): $\mu$ and $\epsilon_{\mathcal{N}}\delta\mu$; $8\pi G = 1$ | Spin-two, spin-one and corner ambiguities remain part of the prescription; linear $\mu$ is distinct from quadratic $P_L$. |
| Odak–Rignon-Bret–Speziale, 2309.03854v3 | Eqs. (3.26)–(3.28): polarization exchange; Eq. (4.6): mixed scalar condition; $16\pi G = c = 1$ | Generator and spin-two restrictions also occur; Eq. (4.8) imposes the joint area condition. |

The source formulas retain their matter, generator, embedding and corner conditions. The stationary corner route below shares the area normalization and prescribed clock and supplies a complementary interpretation.

*b. Direct action and endpoint check.* For the fixed Gaussian-null patch in Appendix I, the EH current decomposes into the area-polarized scalar term, shear response, and $\delta B_0$. The explicit prescription $I_{\rm rate} = I_{\rm EH} - B_0 + B_\mu$ therefore gives Eq. (14); see Eqs. (I3)–(I7). In geometric units the net shift is $B_\mu - B_0 = -(A_f - A_i)/(16\pi G)$. Fixed endpoint coordinates do not remove its variation. Tangent nonexpansion throughout the fixed interval instead gives $\delta A_f = \delta A_i$, without requiring either variation to vanish individually [Eqs. (I8)–(I9)]. Nonexpansion alone does not remove shear. The appendix displays the remaining response and tests the restrictions with nonzero controls. This calculation supplies a local action result; admissible on-shell source families and the other boundary/corner data retain their stated conditions.

### B. Acceleration-length source representation

Restrict to a phase-space sector consisting of non-expanding null boundaries and variations tangent to that sector:

$$\theta_\xi = 0, \qquad \delta\theta_\xi = 0. \tag{16}$$

Then $\mu_\xi = \widehat{\kappa}_\xi$. Restore physical units by defining the acceleration-valued surface gravity

$$\kappa_\xi \equiv c^2\widehat{\kappa}_\xi. \tag{17}$$

Expressing the null evolution in a prescribed physical clock $\tau$ gives Eq. (14) in the form

$$\Theta_{\kappa}^{(0)} = \frac{c^2}{8\pi G}\int d\tau \int_S \epsilon_S\, \delta\kappa_\xi. \tag{18}$$

For uniform stationary data, write $\vartheta_A = \Theta_A^{(0)}/\Delta\tau$ and $\vartheta_\Pi = \Theta_\kappa^{(0)}/\Delta\tau$. The established $\Pi_\partial = c^2\kappa/(8\pi G)$ gives the direct stress representation

$$\vartheta_A = -\Pi_\partial\, \delta A, \qquad \vartheta_\Pi = \vartheta_A + \delta(A\Pi_\partial) = A\,\delta\Pi_\partial. \tag{19}$$

The generating boundary term is $\Delta\tau\, A\Pi_\partial$; its energy-valued product is the same $U_\partial = A\Pi_\partial$ used in the potential-first route. The stress change is already $\delta\Pi_\partial = -P_L\delta L_\mathrm{s}$, with $P_L = -d\Pi_\partial/dL_\mathrm{s} = \Pi_\partial/L_\mathrm{s}$. Thus the local response per area is pressure, and the uniform-patch response is $\vartheta_\Pi = -AP_L\delta L_\mathrm{s} = -F_L\delta L_\mathrm{s}$, $F_L = AP_L$. They may be identified before naming $U_\partial$, but not independently of that boundary structure. The nontrivial input is the matching action, clock, polarization and admitted source space, not the chain rule. The potential's negative length derivative already defines a generalized force; this calculation establishes its specific action-source role. The identity $F_L = M_\partial\kappa$ is not a second independent dynamical test. The exact differential $dU_\partial$ is distinct from either selected, generally nonexact, scalar one-form. Their difference is $\delta U_\partial$, and their field-space exterior derivatives agree. On the positive thermal branch, with the matching Einstein entropy and Hawking–Unruh temperature, the same exchange becomes $-T\delta S + \delta(TS) = S\delta T$; it does not change the conventional total-mass first law or require thermal input to the classical response. On either nonzero signed branch change the rate coordinate directly

$$L_\mathrm{s} = \frac{c^2}{\kappa_\xi}, \qquad \delta\kappa_\xi = -\frac{c^2}{L_\mathrm{s}^2}\,\delta L_\mathrm{s}. \tag{20}$$

Substitution into Eq. (18) gives the pressure coefficient

$$\boxed{P_L = -\frac{d\Pi_\partial}{dL_\mathrm{s}} = \frac{\Pi_\partial}{L_\mathrm{s}} = \frac{c^4}{8\pi G L_\mathrm{s}^2} = \frac{\kappa_\xi^2}{8\pi G}.} \tag{21}$$

The same selected one-form therefore displays the local pressure density explicitly:

$$\boxed{\begin{aligned}\Theta_L^{(0)} &= -\frac{c^4}{8\pi G}\int d\tau \int_S \frac{\epsilon_S}{L_\mathrm{s}^2}\,\delta L_\mathrm{s} \\ &= -\int d\tau \int_S \epsilon_S P_L\,\delta L_\mathrm{s}.\end{aligned}} \tag{22}$$

For a source uniform on a finite cut,

$$F_L \equiv \int_S P_L\, dA = AP_L, \qquad \Theta_L^{(0)} = -\int d\tau\, F_L\delta L_\mathrm{s}. \tag{23}$$

For uniform sources, define the volume-valued source one-form

$$\vartheta_V^{(\delta)} \equiv A\,\delta L_\mathrm{s}. \tag{24}$$

Then

$$\boxed{\Theta_L^{(0)} = -\int d\tau\, P_L\vartheta_V^{(\delta)}.} \tag{25}$$

This is simply $F_L\,\delta L_\mathrm{s} = P_L A\,\delta L_\mathrm{s}$. The notation $\vartheta_V^{(\delta)}$ denotes a one-form, not the variation of a global volume function on the independent source space. At fixed area it is exact. The canonical result identifies the normal-scale force per area; its conversion to a spatial traction additionally requires an operational displacement and clock prescription.

For the selected mixed action, orientation, cuts, counterterm prescription, and fixed remaining sources, the local-in-time Hamilton–Jacobi relation is

$$F_L(\tau) = -\frac{\delta I_\mathrm{os}}{\delta L_\mathrm{s}(\tau)}. \tag{26}$$

If the source is constant through a stationary interval $\Delta\tau$, then $-(\partial I_{\rm os}/\partial L_{\rm s})_{\Delta\tau} = \Delta\tau F_L$. Here canonical response means the coefficient of an allowed normalized length-source variation in the selected action, equivalently the displayed functional derivative when the admitted on-shell family exists. This role does not require $U_\partial$ to equal the entire Hamiltonian. The complete Hamiltonian is defined by its evolution, boundaries and reference prescription; $P_L$ here is a boundary-response density.

Operationally, the force derivative compares nearby boundary-value problems, each with its own prescribed value of the rate source, while the remaining sources are controlled. It is not a variation that both fixes and changes the same source within one problem. The field-space variation preserves the stated nullness, cut identification, and nonexpanding conditions. Fixing the physical clock prescription does not force $\delta\kappa = 0$, because the metric connection may vary. No existence theorem for arbitrary independent boundary assignments is implied.

In ordinary Hamilton–Jacobi theory an endpoint coordinate derivative of the on-shell action is a momentum. Here $\delta I_{\rm os} = -\int d\tau F_L \delta L_{\rm s}$ makes the local functional derivative a force: action divided by time and length has force units. The constant-source derivative across an interval instead has momentum units. The full variational problem retains its other response and endpoint terms, so this local statement does not imply a globally separate scale-energy function.

The corresponding extended-source two-form is

$$\Omega_L^{(0)} = \delta\Theta_L^{(0)} = \int d\tau \int_S \delta L_{\rm s} \wedge \delta(\epsilon_S P_L). \tag{27}$$

The boundary Legendre transform from Eq. (12) to Eq. (14) is the genuine change of polarization. The subsequent map $\kappa \mapsto c^2/\kappa$ is an invertible source coordinate change inside the rate-source polarization, not a second Legendre transform. For a uniform stationary interval,

$$\frac{\Omega^{(0)}}{\Delta\tau} = \delta A \wedge \delta\Pi_\partial = \delta L_{\rm s} \wedge \delta F_L. \tag{28}$$

These describe one scalar canonical pair in different polarizations; they are not two additional canonical degrees of freedom. The local area measure is retained: pressure is a response density, not an extra independent partner of length.

*a. Why acceleration length is the preferred work coordinate.* The selection is conditional and geometric. Require a length made only from the prescribed acceleration-valued rate $\kappa$ and $c$, without another dimensional scale or additional boundary variable. Dimensional analysis fixes its form to $Cc^2/\kappa$, and the operational Rindler distance fixes $C = 1$ on the positive branch. This identifies $L_{\rm s}$ as the normalized length coordinate for the new force and pressure responses. It is also the reduced inverse-temperature scale after the thermal identification. In the specified equilibrium FRW family it equals the areal radius; in Schwarzschild it is twice that radius.

This is not uniqueness among all possible coordinates. For any smooth invertible $q = f(L_{\rm s})$, the transformed coefficient $F_q = F_L\, dL_{\rm s}/dq$ preserves $F_q\, dq = F_L\, dL_{\rm s}$. For example, $q = 2L_{\rm s}$ gives $F_q = F_L/2$: the physics remains equivalent, but $q$ is no longer the normalized Rindler distance. Surface gravity remains the regular rate coordinate, including at $\kappa = 0$. The action establishes the chosen source response; the geometric length conditions select its direct force–displacement presentation. The identity $F_L L_{\rm s} = U_\partial$ is a consequence, not an independent uniqueness test. The radius representation must likewise retain the additional ratio variation when $L_{\rm s}/R$ is not fixed; see Sec. V.

The result may be summarized as follows.

> *Non-expanding null-boundary source-response proposition.* Let $\mathcal{N}$ be a four-dimensional non-expanding Einstein null boundary equipped with a prescribed physical evolution vector, fixed relational cuts, and variations tangent to the non-expanding sector. Assume that matter boundary sources and omitted symplectic flux are controlled separately. On either branch $\kappa_\xi \neq 0$, the signed acceleration length $L_{\rm s} = c^2/\kappa_\xi = 1/\widehat{\kappa}_\xi$ is an admissible reparameterization of the covariant mixed spin-zero source. Its Hamilton–Jacobi response density is $P_L = \kappa_\xi^2/(8\pi G)$, and a uniform finite cut carries $F_L = AP_L$.

The hypotheses are substantive. On a generic expanding null boundary the source is $\mu$, so after defining $\mu_{\rm acc} = c^2\mu$ the clean reciprocal coordinate is

$$L_\mu = \frac{c^2}{\mu_{\rm acc}}, \tag{29}$$

not $c^2/\kappa$ alone. At $\kappa = 0$ the reciprocal chart is singular, while the parent rate is regular. Indeed,

$$-P_L\delta L_{\rm s} = -\frac{\kappa_\xi^2}{8\pi G}\delta\left(\frac{c^2}{\kappa_\xi}\right) = \frac{c^2}{8\pi G}\delta\kappa_\xi. \tag{30}$$

$P_L \to 0$ at $\kappa \to 0$ does not make the parent variation vanish; it means that the $L$ chart has reached infinity.

*b. The scalar response when expansion is retained.* There is a useful extension at the level of the same selected scalar one-form, not a complete moving-horizon theorem. With the clock measure fixed as above and $\mu_{\rm acc} = c^2\mu = \kappa + c^2\theta/2$, the scalar part is

$$\Theta_\mu^{(0)} = -\int d\tau \int_S \epsilon_S P_L\, \delta L_{\rm s} + \int d\tau \int_S \epsilon_S \frac{c^4}{16\pi G}\, \delta\theta. \tag{31}$$

This follows by substituting $\delta\mu = \delta\kappa/c^2 + \delta\theta/2$ into Eq. (14). The original coefficient survives as a partial scalar response; the independent expansion term remains. At fixed expansion it drops out, provided the remaining boundary and constraint conditions admit that variation. Alternatively, $L_\mu = 1/\mu$ absorbs the complete scalar source and gives $P_\mu = c^4/(8\pi G L_\mu^2)$, but $L_\mu$ is not generally $L_{\rm s}$. Neither representation removes vector, shear, matter, or corner terms. It therefore does not establish that a non-null FRW apparent-horizon worldtube has the same one-source response.

### C. Stationary area–boost source transformation

The selected null-source calculation is the principal action-level justification. The stationary area–boost route gives a complementary geometric check. Both use Einstein area normalization, the same physical clock and the stationary boost–surface-gravity relation. Their agreement is not an independent observational confirmation.

The stationary corner sector supplies a complementary check of the same response. The normal boost is parameterized using the same physical clock and acceleration length as the rest of this manuscript. Einstein gravity assigns the relative boost angle $\eta$ of the boundary normals the corner momentum

$$J_\eta = \frac{c^3 A}{8\pi G} \tag{32}$$

The charge is the coefficient of a dimensionless relative Lorentz boost of the normal frames at the corner. It has action units and is not the black hole's rotational angular momentum; changing the area changes this charge. Calling it a charge therefore does not assert conservation during arbitrary horizon evolution [15–17]. For a uniform stationary interval of prescribed duration $\Delta\tau$,

$$\eta = \frac{\kappa_\xi \Delta\tau}{c} = \frac{c\Delta\tau}{L_{\rm s}}. \tag{33}$$

With $\Delta\tau$ fixed, and choosing the canonical orientation consistently with the null-source response,

$$J_\eta \delta\eta = -\Delta\tau F_L \delta L_{\rm s}, \qquad \Omega_{\rm corner} = \Delta\tau\, \delta L_{\rm s} \wedge \delta F_L. \tag{34}$$

The momentum conjugate to $L_{\rm s}$ in this stationary edge-mode sector is

$$p_L = J_\eta \frac{\partial\eta}{\partial L_{\rm s}} = -\frac{c\Delta\tau J_\eta}{L_{\rm s}^2} = -\Delta\tau \frac{A\kappa^2}{8\pi G} = -\Delta\tau F_L. \tag{35}$$

The quantity $p_L$ has momentum units; $F_L$ is the response per unit boundary time. It is distinct from the pressure $P_L$. A force appears because the boost angle is a rate integrated over time and $L_{\rm s}$ is the reciprocal of that rate.

Equations (18) and (34) locate the same coefficient in the null and stationary corner representations of Einstein boundary mechanics. Therefore the central canonical algebra is not contingent on a thermodynamic analogy. Its remaining conditionality concerns the selected boundary ensemble, clock, cuts, and allowed counterterms, not the chain-rule calculation itself.

The compact dictionary of Part I follows from the same pair:

$$\omega_s = \frac{\kappa}{c}, \qquad U_\partial = \omega_s J_\eta, \qquad \boxed{U_\partial L_{\rm s} = cJ_\eta = \frac{Ac^4}{8\pi G}.} \tag{36}$$

Equivalently, $U_\partial = J_\eta/(L_{\rm s}/c)$: an action-valued boost generator divided by the time for one unit of normal boost gives the energy for that evolution. The coefficient contains $c^3$, while the rate $c/L_{\rm s}$ supplies the fourth power of $c$. This interpretation uses the framework's acceleration length directly. It is not a new independently conserved energy, and an oriented corner action must not be confused with the selected canonical one-form or the complete Hamiltonian variation.

The product $U_\partial L_s/c$ is the boost charge itself. At fixed area, energy and reciprocal-rate length vary inversely. Under a constant physical-clock rescaling the product is unchanged, although $U_\partial$, $L_s$, and $P_L$ separately change. The area and rate legs read $\omega_s dJ_\eta = \Pi_\partial dA$ and $J_\eta d\omega_s = -F_L dL_s$. Thermality is unnecessary for these identities. Only with the subsequent Hawking–Unruh identification do they become $TdS$ and $SdT$. For $\kappa < 0$ the thermal dictionary uses magnitudes; the mechanical identity above retains signed $U_\partial$ and $L_s$.

The integrated stationary relation $\eta = \kappa\Delta\tau/c$ uses the prescribed stationary normal flow. A generic moving cosmological screen has normal-connection and transport data not captured by its elementary velocity rapidity. The area–boost pair alone does not establish that integrated relation with an arbitrarily substituted dynamical surface gravity.

### D. Canonical radius representation of the spherical scalar sector

Keep the clock, interval, cuts, orientation, and scalar-sector prescriptions of the preceding derivation. On uniform spherical cuts, $(R, \alpha_R)$ are coordinates on the same two-dimensional source space as $(A, L_s)$. Since $U_\partial = RF_R$ and $F_R = c^4/(2G\alpha_R)$, the rate-source potential becomes

$$\begin{aligned}\frac{\Theta^{(0)}}{\Delta\tau} &= \frac{Ac^2}{8\pi G}\,\delta\kappa = -F_R\,\delta R - U_\partial\frac{\delta\alpha_R}{\alpha_R}\\ &= R\,\delta F_R - F_R\,\delta R.\end{aligned} \tag{37}$$

Consequently,

$$\boxed{\frac{\Omega^{(0)}}{\Delta\tau} = 2\,\delta R\wedge\delta F_R, \qquad \frac{\Theta^{(0)}_A}{\Delta\tau} = -\Pi_\partial\,\delta A = -2F_R\,\delta R.} \tag{38}$$

Thus the full scalar pair is represented by $R$ and $2F_R$, with momentum $-2\Delta\tau F_R$ in the displayed area polarization. The potentials differ by $\delta U_\partial$ per unit prescribed time. Calling $F_R$ alone the full radius momentum would miss a factor of two. It is the coefficient of the scale-work leg when $\delta\alpha_R = 0$; the area response is twice that coefficient. On a fixed-$\alpha_R$ one-parameter family, the pulled-back two-form vanishes, as it must. No extra canonical degree of freedom is created.

In particular, Schwarzschild has $\delta\alpha_R = 0$ and $\alpha_R = 2$. Equation (37) becomes $\Theta^{(0)}/\Delta\tau = -M\kappa\,\delta R_H = -P_R\,\delta V_H$. This is the original selected response restricted to the stated Schwarzschild variations; the same clock and boundary conditions apply.

One can also define an area-equivalent $R = \sqrt{A/(4\pi)}$ for a finite nonspherical patch and make the same algebraic transformation. This does not supply a preferred center, a spherical volume, or a physical Misner–Sharp mass. Those interpretations require spherical geometry and, for $M_R$ to be its actual enclosed mass, marginality.

The corner calculation gives the same result directly: $J_\eta\delta\eta/\Delta\tau = Ac^2\delta\kappa/(8\pi G)$ at fixed interval, so substituting $\kappa = c^2/(\alpha_R R)$ reproduces Eq. (37). This is a radius representation of the established area–rate and area–boost structures, not an independent canonical foundation. Its application to a generic expanding cosmological screen is not proved by this non-expanding-null argument; the Cai–Kim realization below remains a separate prescription.

## III. CLOCK, GAUGE, AND THE STANDARD FIRST LAW

### A. Auxiliary normal versus physical clock

Surface gravity is a rate relative to a normal evolution. To separate an auxiliary null-frame choice from a physical clock, write

$$\xi^a = f\ell^a, \qquad \widehat{\kappa}_\xi = f\widehat{\kappa}_\ell + \ell[f]. \tag{39}$$

Under the simultaneous change

$$\ell^a \longrightarrow e^\sigma\ell^a, \qquad f \longrightarrow e^{-\sigma}f, \tag{40}$$

the physical vector $\xi^a$ is unchanged, and direct substitution shows that $\widehat{\kappa}_\xi$, $\kappa_\xi$, and $L_s$ are unchanged. The construction is therefore independent of the auxiliary null representative for a fixed physical normal flow [14].

Changing the physical clock is different. If $\xi^a \to \alpha\xi^a$, then

$$\widehat{\kappa}_{\alpha\xi} = \alpha\widehat{\kappa}_\xi + \xi[\alpha], \qquad \kappa_{\alpha\xi} = \alpha\kappa_\xi + c^2\xi[\alpha]. \tag{41}$$

For constant $\alpha$,

$$L_{\rm s} \longrightarrow \frac{L_{\rm s}}{\alpha}, \qquad P_L \longrightarrow \alpha^2 P_L. \tag{42}$$

The area and intrinsic two-geometry do not change. $P_L$ is a well-defined response after a clock prescription has been fixed, but is not an intrinsic scalar of the cut alone. This dependence is inherited, not a new weakness of the pressure: the same is true of surface gravity, energy, temperature, membrane stress, and Hamiltonian charge, which are also defined relative to the selected evolution vector.

Rindler, Schwarzschild, and spherical dynamics use different clocks but perform the same geometric task. Rindler uses boost flow normalized by the accelerated observer's proper time; Schwarzschild uses Killing time normalized at infinity; spherical dynamics uses the Kodama flow or the Cai–Kim instantaneous thermal calibration. What is common is a prescribed evolution in the two-dimensional normal plane, not a universal observer-independent accelerometer reading.

For a proper-gauge direction $X$ preserving the selected boundary conditions, the source and response descend individually when

$$X[L_{\rm s}] = 0, \qquad X[P_L\epsilon_S] = 0, \tag{43}$$

and the scalar two-form defines a separate reduced sector when it is basic,

$$\iota_X\Omega_L^{(0)} = 0, \qquad \mathcal{L}_X\Omega_L^{(0)} = 0. \tag{44}$$

These conditions hold in the stationary edge-mode sector when clock and cuts are fixed relationally or retained as edge data and only source-preserving degeneracies are quotiented. They fail if arbitrary physical-clock rescalings are declared gauge.

### B. Why the standard first law has no additional work term

The usual stationary first law computes the Hamiltonian variation of a specified horizon evolution. On a fixed cut the covariant-phase-space expression has the schematic form

$$\delta H_\xi = \int_S \left(\delta Q_\xi - \xi\cdot\Theta - Q_{\delta\xi}\right), \tag{45}$$

with additional transport terms when the cut moves [21–23]. "Fixed generator" means $\delta\xi^a = 0$ in field space; it does not generally mean $\delta\kappa_\xi = 0$. The complete expression cancels or reallocates pieces caused only by changing the measuring generator, leaving the familiar area channel.

Hayward, Mukohyama, and Ashworth derived the spherical Kodama–Noether charge $\oint Q_K = A\widehat{\kappa}_H/(8\pi)$ in geometric units and explicitly questioned the usual suppression of $\delta\widehat{\kappa}$ [19]. Parattu, Majhi, and Padmanabhan showed that the two pieces of the Einstein surface-Hamiltonian variation become $T\delta S$ and $S\delta T$ in thermodynamically conjugate variables [20]. The acceleration-length representation gives the latter a direct mechanical coordinate on the positive thermodynamic branch:

$$S\,\delta T = -F_L\delta L_\kappa. \tag{46}$$

The absence of $S\,dT$ from the standard mass first law is therefore not a missing additive energy. It is the consequence of a different boundary-value problem: the usual law varies the Hamiltonian mass for a specified evolution, whereas Eq. (46) resolves the boundary potential $U_\partial = TS$ when the normal-rate source is varied.

## IV. BOUNDARY EQUATION, PULLBACK, AND MECHANICAL–THERMAL SCALE

### A. One potential and two pressures

For a stationary uniform cut, the established stress obeys $\kappa_{\rm E}\Pi_\partial = \kappa/c^2 = 1/L_{\rm s}$. The chosen Einstein boundary functional, with no additional area-dependent integration term, therefore gives

$$U_\partial(A, L_{\rm s}) = \Pi_\partial A = \frac{A}{\kappa_{\rm E}L_{\rm s}} = \mathcal{T}_G\frac{A}{L_{\rm s}}, \qquad \mathcal{T}_G = \frac{c^4}{8\pi G}. \tag{47}$$

Its first derivatives are

$$\Pi_\partial \equiv \left(\frac{\partial U_\partial}{\partial A}\right)_{L_\mathrm{s}} = \frac{c^4}{8\pi G L_\mathrm{s}} = \frac{c^2\kappa}{8\pi G}, \tag{48}$$

$$F_L \equiv -\left(\frac{\partial U_\partial}{\partial L_\mathrm{s}}\right)_A = \frac{c^4 A}{8\pi G L_\mathrm{s}^2}, \qquad P_L = \frac{F_L}{A}. \tag{49}$$

The quantities are dimensionally and variationally different. $\Pi_\partial$ is energy per area or force per length, the pressure/tension of a two-dimensional fluid. $P_L$ is force per transverse area, the normal scale pressure.

The pressure interpretation of $\Pi_\partial$ is independently established in the Damour–Navier–Stokes equation and membrane paradigm, where the tangential momentum balance contains the pressure-gradient term $-D_A\Pi_\partial$ [24–27]. Depending on orientation and stress convention, the same magnitude is called pressure or tension. The pressure interpretation of $P_L$ is instead fixed by Eq. (25). The relation between them is the Hessian identity

$$-\left(\frac{\partial \Pi_\partial}{\partial L_\mathrm{s}}\right)_A = \left(\frac{\partial F_L}{\partial A}\right)_{L_\mathrm{s}} = P_L = \frac{\Pi_\partial}{L_\mathrm{s}}. \tag{50}$$

It is useful to call $\Pi_\partial$ the *surface pressure/tension* and $P_L$ the *normal scale pressure*.

### B. Displacement, force per area, and volume

The generalized force $F_L = -(\partial U_\partial/\partial L_\mathrm{s})_A$ and its pressure $P_L = F_L/A$ are defined before any volume is introduced. For a specified displacement coordinate $x$, holding area and the remaining sources fixed gives

$$F_L\, dL_\mathrm{s} = F_x^{(L)}\, dx, \qquad F_x^{(L)} = F_L \left(\frac{\partial L_\mathrm{s}}{\partial x}\right)_A. \tag{51}$$

If $dV = A\, dx$, the coefficient of that volume work is $P_x = P_L(\partial L_\mathrm{s}/\partial x)_A$. Thus $P_L$ itself is the pressure associated with $dV$ only when the chosen displacement has $dx = dL_\mathrm{s}$. A proper spatial displacement is not supplied by the units of $L_\mathrm{s}$ alone. Rindler provides an exact proper-distance realization; other sectors specify their own geometric dictionary. For Schwarzschild, $dL_\kappa = 2dR_H$ gives $P_R = 2P_L$; equilibrium FRW has $dL_\kappa = dR_A$ and $P_R = P_L$. The work is invariant under this joint change of displacement and response coefficient.

On a physical family both area and scale usually vary. Its full pullback is

$$\frac{dU_\partial}{dx} = \Pi_\partial \frac{dA}{dx} - F_L \frac{dL_\mathrm{s}}{dx}. \tag{52}$$

The second term is the scale-work leg, not generally the negative total derivative. For the positive-branch product $\mathcal{V}_\partial = AL_\kappa$ one instead has

$$d\mathcal{V}_\partial = A\, dL_\kappa + L_\kappa\, dA, \qquad dU_\partial = 2\Pi_\partial\, dA - P_L\, d\mathcal{V}_\partial. \tag{53}$$

This explains why the Euler identity $U_\partial = P_L\mathcal{V}_\partial$ cannot be inserted into the original differential while leaving the area coefficient unchanged. The force–length form avoids that ambiguity entirely.

### C. What the pullback statement means

The word pullback has a precise differential-geometric meaning. Fix a nonzero sign branch $\sigma = \mathrm{sgn}(\kappa)$, let $S$ be a horizon cut, and regard the signed acceleration length as a map

$$L_\mathrm{s} : S \longrightarrow \mathbb{R}_\sigma, \qquad x \longmapsto L_\mathrm{s}(x), \qquad \mathbb{R}_\sigma \equiv \{L : \sigma L > 0\}. \tag{54}$$

On that one-dimensional source branch, define

$$\Pi(L) = \frac{c^4}{8\pi G L}, \qquad d\Pi = -P_L(L)\, dL, \qquad P_L(L) = \frac{c^4}{8\pi G L^2}. \tag{55}$$

The pullback $L_{\rm s}^*(d\Pi)$ is the one-form on $S$ defined at a point $x$ and tangent vector $X \in T_xS$ by

$$\big[L_{\rm s}^*(d\Pi)\big]_x(X) = d\Pi_{L_{\rm s}(x)}\big(\big(dL_{\rm s}\big)_x(X)\big). \tag{56}$$

In coordinates this is simply the chain rule:

$$\boxed{L_{\rm s}^*(d\Pi) = d(\Pi \circ L_{\rm s}) = -P_L\, dL_{\rm s}, \qquad D_A\Pi_\partial = -P_L D_A L_{\rm s}.} \tag{57}$$

On the same fixed-sign branch, $L_\kappa = |L_{\rm s}|$ and

$$\Pi_\partial = \sigma P_L L_\kappa, \qquad D_A\Pi_\partial = -\sigma P_L D_A L_\kappa, \qquad |\Pi_\partial| = P_L L_\kappa, \qquad D_A|\Pi_\partial| = -P_L D_A L_\kappa. \tag{58}$$

Consequently the pressure force in the tangential horizon-fluid equation can be written exactly as

$$-D_A\Pi_\partial = P_L D_A L_{\rm s} = \sigma P_L D_A L_\kappa. \tag{59}$$

The established membrane pressure-gradient one-form is therefore the pullback to the physical cut of the signed source-space response one-form $-P_L dL_{\rm s}$. Equivalently, the stress-magnitude gradient is the pullback of $-P_L dL_\kappa$.

This relation gives $P_L$ an organizing role, but it does not make it ontologically more fundamental than every parent quantity. $\Pi_\partial$ is the regular surface stress that appears directly in the tangential momentum equation, while $P_L$ is its susceptibility with respect to the mechanically privileged reciprocal-rate coordinate. Moreover, on a generic expanding null boundary the still more general parent source is $\mu = \widehat{\kappa} + \theta/2$. The accurate hierarchy is therefore: $\mu$ is the generic null source; $\Pi_\partial$ is the established linear surface stress on the selected non-expanding sector; and $P_L$ is the normal pressure that generates the change of that stress with acceleration length.

Here $D_A$ is the tangential covariant derivative on the cut (its index is not the total area $A$). It measures a spatial stress gradient, whereas $\delta$ compares boundary data in field space. "Pullback" means using $L_{\rm s}(x)$ to evaluate the source-space identity on the cut. This changes neither the membrane equation nor its other shear and momentum terms; it identifies which spatial pattern of the established surface stress is represented by an acceleration-length gradient.

### D. A genuine normal force-balance equation

The same relation enters a normal, rather than tangential, projection of Einstein's equation. Freidel and Yokokura showed that a timelike gravitational screen obeys the non-equilibrium equations of a viscous bubble, including a generalized Young–Laplace equation [28]. Their screen tension is defined by the screen normal-rate variable $\gamma_t$. In their geometric-unit conventions set $\kappa_{\rm scr} = \gamma_t$ and define the corresponding $L_{\rm s,scr} = 1/\kappa_{\rm scr}$ and $P_{L,\rm scr} = \kappa_{\rm scr}^2/(8\pi G)$. Suppressing these screen subscripts only in this subsection, their orientation gives

$$\gamma = -\Pi_\partial = -P_L L_{\rm s}. \tag{60}$$

For a static screen the normal momentum balance is

$$0 = \Delta p + \gamma H = \Delta p - \Pi_\partial H, \tag{61}$$

where $H$ is the mean curvature of the cut in the spatial screen and $\Delta p = p_{\rm out} - p_{\rm in}$ includes both matter and the appropriate intrinsic three-curvature contribution. Substitution of Eq. (60) gives the exact signed form

$$\boxed{\Delta p - P_L L_{\rm s} H = 0.} \tag{62}$$

In the positive-$\kappa$ orientation this is $\Delta p - P_L L_\kappa H = 0$; reversing the screen orientation reverses the signed stress and pressure-jump convention together. $P_L$ enters a genuine normal force balance: its pressure units are converted into curvature pressure by the dimensionless factor $L_{\rm s}H$, or by $L_\kappa H$ in positive magnitudes. In general the traction is $\Delta p = P_L L_{\rm s} H$, not $P_L$ alone. Identifying this screen rate with a particular horizon surface gravity requires a matching flow, lapse, and normalization; it is not automatic.

The full dynamical screen equation, in the geometric units and notation of Ref. [28], is

$$-(D_{\boldsymbol{t}} + \theta_{\boldsymbol{t}})\theta_{\boldsymbol{n}} = 8\pi G\big[\Delta p - P_L L_{\rm s} H\big] - \frac{1}{2}\widetilde{\Theta}_{\boldsymbol{t}} : \Theta_{\boldsymbol{n}} - \varrho\,\omega^2 - \Delta\varrho. \tag{63}$$

Here the additional terms encode shear/viscous, normal-connection, and lapse-gradient effects. In a spherically symmetric, irrotational, constant-lapse reduction,

$$-\left(D_{\boldsymbol{t}}+\frac{3}{4}\theta_{\boldsymbol{t}}\right)\frac{\theta_{\boldsymbol{n}}}{8\pi G}=\Delta p-P_L L_{\mathrm{s}} H. \tag{64}$$

Equations (62)–(64) give the normal force-balance extension under this screen-rate dictionary. They are exact acceleration-length rewritings of an established gravitational-screen equation, not an additional field equation. They also clarify why no three-dimensional Navier–Stokes equation for $P_L$ is required. The Damour equation governs tangential two-dimensional fluid momentum; the Young–Laplace/normal constraint governs normal screen motion. $P_L$ links the two through $\Pi_\partial = P_L L_{\mathrm{s}}$, or through $|\Pi_\partial| = P_L L_\kappa$ in positive magnitudes.

### E. Mechanical–thermal bridge and entropy density

For the thermodynamic magnitudes in this subsection, choose the positive-$\kappa$ orientation. Then $L_{\mathrm{s}} = L_\kappa$, $\Pi_\partial = |\Pi_\partial|$, and $U_\partial = |U_\partial| = TS$. Reversing orientation leaves $T$, $P_L$, $L_\kappa$, and all magnitudes unchanged while reversing the signed stress and work form, as summarized in Appendix A. Einstein entropy and the Hawking–Unruh temperature are

$$S=\frac{k_B c^3 A}{4G\hbar}=\frac{k_B A}{4\ell_{\mathrm{P}}^2},\qquad \ell_{\mathrm{P}}^2\equiv\frac{G\hbar}{c^3}, \tag{65}$$

$$T=\frac{\hbar|\kappa|}{2\pi k_B c}=\frac{\hbar c}{2\pi k_B L_\kappa},\qquad L_\kappa=\frac{c^2}{|\kappa|}. \tag{66}$$

Equation (66) identifies $L_\kappa$ as a *mechanical–thermal bridge scale*. Its Euclidean meaning is strongest for a nonextremal stationary horizon admitting a regular Hawking–Unruh/KMS state. Wick rotation is the analytic continuation of the selected Lorentzian time,

$$t=-i\tau_E, \tag{67}$$

not a physical evolution of the spacetime. In the two-plane normal to a horizon cut, the local Lorentzian and Euclidean metrics take the Rindler forms

$$ds^2_{\perp,L}\simeq-\left(\frac{|\kappa|\rho}{c^2}\right)^2 c^2dt^2+d\rho^2,\qquad ds^2_{\perp,E}=d\rho^2+\rho^2d\vartheta^2,\qquad \vartheta=\frac{|\kappa|\tau_E}{c}. \tag{68}$$

The Lorentzian horizon is at $\rho = 0$. In the Euclidean normal two-plane this is the rotation fixed point; in the full spacetime the corresponding fixed-point set is the codimension-two horizon cut, not a single spacetime point. Smoothness at $\rho = 0$ requires $\vartheta \sim \vartheta + 2\pi$. Any other period produces a conical defect at the horizon. Therefore

$$\Delta\tau_E=\frac{2\pi c}{|\kappa|},\qquad c\Delta\tau_E=2\pi L_\kappa. \tag{69}$$

The reason this periodicity is thermal is standard quantum statistical mechanics. The canonical partition function $Z(\beta) = \mathrm{Tr}\, e^{-\beta H}$ is represented by evolution through imaginary time $\hbar\beta$ with the endpoints identified. Bosonic correlators are periodic and fermionic correlators antiperiodic around that imaginary-time circle; the Lorentzian KMS condition is the operator-algebraic equilibrium statement of the same analyticity and periodicity. For the Minkowski vacuum restricted to a Rindler wedge, the Bisognano–Wichmann result gives the exact KMS property with respect to boosts, while an accelerated detector obeys the corresponding Unruh detailed balance. For a regular stationary black-hole Euclidean saddle, the same smoothness condition fixes the Hawking temperature [7, 33–35].

Writing $\beta = (k_B T)^{-1}$, the bosonic Matsubara frequencies are

$$\omega_n^{\mathrm{E}}=\frac{2\pi n}{\hbar\beta}. \tag{70}$$

Equations (66) and (69) give

$$\Omega_\kappa\equiv\frac{|\kappa|}{c}=\frac{2\pi k_B T}{\hbar}=\omega_1^{\mathrm{E}},\qquad L_\kappa=\frac{c}{\Omega_\kappa}=\frac{c}{\omega_1^{\mathrm{E}}}. \tag{71}$$

The standard content is $\hbar\beta = 2\pi c/|\kappa|$ together with the Matsubara spectrum. Equation (71) is an exact corollary, familiar in natural units as $1/|\kappa| = \beta/(2\pi)$, but it is not usually foregrounded as a horizon-mechanical coordinate. The new structural use here is that the same reduced Euclidean thermal length is also the source coordinate whose Einstein boundary response is $P_L$. It should not be confused with the Lorentzian wavelength of a typical emitted Hawking quantum. In Rindler spacetime $L_\kappa$ is literally the radius of the Euclidean orbit of the selected accelerated observer; for a globally normalized black-hole clock it is the Euclidean time period in light-travel units divided by $2\pi$, not generally a proper radial distance [36].

The two laws carried by the same scale are

$$\boxed{P_L(L_\kappa) = \frac{c^4}{8\pi G L_\kappa^2}, \qquad k_B T(L_\kappa) = \frac{\hbar c}{2\pi L_\kappa}.} \tag{72}$$

The first is classical gravitational boundary mechanics and scales as $L_\kappa^{-2}$; the second is semiclassical quantum-field temperature and scales as $L_\kappa^{-1}$. Their ratio is independent of the clock scale:

$$\boxed{\frac{k_B T}{P_L L_\kappa} = 4\ell_{\mathrm{P}}^2.} \tag{73}$$

The numerical entropy-density relation could already be written in the established membrane form $\Pi_\partial/T = S/A$. The acceleration-length factorization

$$\Pi_\partial = P_L L_\kappa, \qquad P_L = -\left(\frac{\partial|\Pi_\partial|}{\partial L_\kappa}\right) \tag{74}$$

does not alter that coefficient; it supplies the missing mechanical resolution. It identifies a normal force $F_L = AP_L$, a displacement $L_\kappa$, and pressure work $F_L\, dL_\kappa = P_L\, \vartheta_V$. It also makes the same factorization responsible for the Schwarzschild scale product and the Cai–Kim FRW work term. The entropy density can therefore be read macroscopically as

$$\frac{S}{A} = \frac{P_L L_\kappa}{T} : \qquad \text{mechanical boundary-energy density divided by temperature.} \tag{75}$$

This is the interpretive gain beyond substituting the established surface tension into the Hawking relation.

The boundary potential is separately homogeneous of degree one in $A$ and degree minus one in $L_\kappa$:

$$A\left(\frac{\partial U_\partial}{\partial A}\right)_{L_\kappa} = -L_\kappa\left(\frac{\partial U_\partial}{\partial L_\kappa}\right)_A = U_\partial. \tag{76}$$

Its integrated representations form the Euler chain

$$\boxed{U_\partial = TS = \Pi_\partial A = P_L(AL_\kappa) = F_L L_\kappa.} \tag{77}$$

Combining Eqs. (73) and (77) gives

$$\frac{S}{A} = \frac{P_L L_\kappa}{T} = \frac{k_B}{4\ell_{\mathrm{P}}^2}. \tag{78}$$

As $|\kappa| \to 0$, both $P_L L_\kappa = c^2|\kappa|/(8\pi G)$ and $T = \hbar|\kappa|/(2\pi k_B c)$ vanish linearly, so their ratio retains the finite branchwise limit in Eq. (78). For a regular finite-area Einstein horizon, $TS/A$ can therefore vanish while $S/A$ remains finite. The compatibility of zero temperature with finite entropy is familiar; the pressure–length factorization makes the matching mechanical and thermal scalings explicit. At the exact zero, the reciprocal chart is unavailable and the statement is understood as a limit from either nonzero branch.

The factor $1/4$ is fixed by the ratio of the Einstein boundary coefficient $1/(8\pi G)$ to the Hawking–Unruh coefficient $1/(2\pi)$; surface gravity and clock normalization cancel. This is not a purely general-relativistic derivation because the temperature contains $\hbar$. It is a complete macroscopic boundary-mechanical reconstruction of the Einstein-horizon entropy density at the interface of classical gravitational action and semiclassical quantum-field temperature. It is a consistency reconstruction using the Einstein entropy law already stated in Eq. (65). If one instead starts with $T\, dS = \Pi_\partial\, dA$, integration determines the area term plus an additive constant $S_0$; the Euler equality $U_\partial = TS$ uses the conventional $S_0 = 0$ normalization. No microscopic state count or independent entropy functional is derived.

Ordinary state counting is not a prerequisite for treating this as real entropy. Wald emphasizes that the classical and semiclassical evidence supports black-hole entropy while "it does not seem obvious that black hole entropy corresponds to counting states in any usual sense" [8, 9]. One possible macroscopic informational interpretation is a horizon supplies an observer- or evolution-relative partition between accessible and inaccessible data; boundary mechanics fixes the energy density assigned to that partition; and temperature converts it into entropy. The claim is not that $S$ inventories every detailed fact in the hidden region. Such a reading would relate it to information unavailable to, or thermodynamically indistinguishable within, the selected exterior description. The present equations do not select an observable algebra, coarse-graining map, or statistical entropy whose equality to the area law has been proved.

For a black-hole event horizon the partition is causal for the exterior observer. For an eternally accelerated Rindler observer it is causal and permanent for that observer but remains observer-relative because inertial observers do not share it. A generic FRW apparent horizon is a quasi-local marginal screen and need not be a permanent causal barrier; exact de Sitter is the special case in which the apparent and cosmological event horizons coincide.

At fixed area the differential identity is

$$-S\,dT = F_L\,dL_\kappa = P_L\,\vartheta_V. \tag{79}$$

Gravitational cooling and outward scale work are the same canonical variation in this ensemble. This does not by itself specify an extractable-work protocol.

The geometric product identities remain $U_\partial = P_L A L_\kappa = TS$ and, for the fixed-ratio spherical families, $U_\partial = 3P_R V$. They do not require an effective particle count or a gas model. Define $N_S \equiv S/k_B$ as dimensionless entropy.

*a. Exact inverse-energy scale and entropy displacement.* The boundary potential also defines a quantum inverse-energy scale. Set

$$M_\partial \equiv \frac{U_\partial}{c^2}, \qquad \bar{\lambda}_\partial \equiv \frac{\hbar}{M_\partial c} = \frac{\hbar c}{U_\partial}. \tag{80}$$

For a particle whose $M_\partial$ is invariant rest mass, this is the ordinary reduced Compton wavelength. Here $M_\partial$ is instead the mass equivalent of a clock-normalized boundary potential, not generally particle rest mass, ADM mass, or source mass. The more precise name is therefore *boundary Compton-equivalent length.*

The ratio raised naturally by the boundary first law has a direct thermodynamic meaning. At fixed $U_\partial$,

$$\frac{T}{F_L} = \left(\frac{\partial L_\kappa}{\partial S}\right)_{U_\partial} = \frac{L_\kappa}{S} = \frac{\bar{\lambda}_\partial}{2\pi k_B}, \tag{81}$$

so $T/F_L$ is length per entropy, while

$$\boxed{\frac{k_B T}{F_L} = \frac{L_\kappa}{N_S} = \frac{\bar{\lambda}_\partial}{2\pi}.} \tag{82}$$

The quantity $k_B T/F_L$ is the displacement over which the boundary force does one thermal energy $k_B T$ of work. Equivalently,

$$\boxed{2\pi L_\kappa = N_S \bar{\lambda}_\partial, \qquad F_L \bar{\lambda}_\partial = 2\pi k_B T.} \tag{83}$$

At fixed $U_\partial$, Eq. (81) may also be written

$$dS = 2\pi k_B \frac{dL_\kappa}{\bar{\lambda}_\partial}. \tag{84}$$

A displacement by one boundary Compton-equivalent reduced length gives $\Delta S = 2\pi k_B$. Verlinde postulated the analogous relation for the displacement of an external test particle relative to a holographic screen, motivated by Bekenstein's lowering argument, and chose the $2\pi$ normalization so that it combines with Unruh temperature to give $F = ma$ [3]. Equation (84) has the same mathematical form but a different status and meaning: $\Delta x$ is replaced by the boundary source displacement $dL_\kappa$, the particle mass is replaced by the boundary mass equivalent, and the coefficient follows from the Einstein boundary potential and its Hamilton–Jacobi force. This supports a genuine horizon-boundary origin for the displacement law but does not establish the broader claim that all Newtonian gravity is an entropic force or justify arbitrary holographic screens and equipartition.

The integrated relation is

$$\frac{S}{k_B} = \frac{2\pi U_\partial L_\kappa}{\hbar c} = \frac{2\pi L_\kappa}{\bar{\lambda}_\partial}. \tag{85}$$

It has the algebraic form of saturation of the Bekenstein entropy bound [2]. For Schwarzschild, $U_\partial = Mc^2/2$ and $L_\kappa = 2R_H$, so the product is the usual $Mc^2R_H$; for Cai–Kim FRW, $U_\partial = E_A$ and $L_\kappa = R_A$. For a generic boundary sector, however, $U_\partial$ need not be the total system energy and $L_\kappa$ need not be an enclosing radius. The result is therefore a horizon-boundary equality of Bekenstein form: it suggests that the normal-flow product $U_\partial L_\kappa$, rather than an enclosing radius alone, is the more general boundary object. It is not a new proof of the Bekenstein bound for arbitrary systems.

The ratio between the quantum and horizon scales is especially simple:

$$\boxed{\frac{\bar{\lambda}_\partial}{L_\kappa} = \frac{2\pi}{N_S} = \frac{8\pi\ell_{\mathrm{P}}^2}{A}.} \tag{86}$$

It is invariant under a constant rescaling of the prescribed clock. These are exact scale relations after the thermal identification. They neither construct microscopic boundary states nor establish a physical phase oscillation. The phase interpretation is retained in Appendix G 3; additional Planck-scale matching conditions are confined to Appendix H. Supporting response functions appear in Appendix D.

## V. STATIONARY REALIZATIONS AND THERMAL–MECHANICAL DUALITY

### A. Rindler: the operational meaning of acceleration length

For an observer with constant proper acceleration $a$ in Minkowski spacetime, introduce coordinates centered on that observer,

$$cT = \left(\frac{c^2}{a} + \xi\right)\sinh\!\left(\frac{at}{c}\right), \qquad X = \left(\frac{c^2}{a} + \xi\right)\cosh\!\left(\frac{at}{c}\right). \tag{87}$$

The observer is at $\xi = 0$ and the Rindler horizon is at $\xi = -c^2/a$. The instantaneous rest slice therefore meets the horizon a proper distance

$$L_\kappa = \frac{c^2}{a} \tag{88}$$

behind the observer. The Lorentzian trajectory satisfies $X^2 - c^2T^2 = L_\kappa^2$, a hyperbola. Under the analytic continuation $t = -i\tau_E$ it becomes

$$cT_E = L_\kappa \sin\!\left(\frac{a\tau_E}{c}\right), \qquad X = L_\kappa \cos\!\left(\frac{a\tau_E}{c}\right), \tag{89}$$

a Euclidean circle of radius $L_\kappa$. The Rindler horizon $\rho = 0$ becomes the origin of this normal Euclidean plane; in four dimensions the transverse horizon plane remains as the fixed-point set. One smooth circuit has $a\Delta\tau_E/c = 2\pi$, reproducing the Unruh period. Rindler therefore makes $L_\kappa$ simultaneously the observer–horizon proper distance, the curvature radius of the accelerated worldline, the Euclidean orbit radius, and the reduced thermal length [7, 35, 36].

Proper time fixes the physical clock and $\kappa = a$. The source proposition and corner action then give

$$\boxed{P_L^{\mathrm{R}} = \frac{a^2}{8\pi G}, \qquad F_L^{\mathrm{R}} = \frac{Aa^2}{8\pi G}, \qquad U_\partial^{\mathrm{R}} = \frac{Aac^2}{8\pi G}.} \tag{90}$$

No source mass, curvature, center, or spherical volume is used. To make the transverse geometry explicit, choose Cartesian coordinates $y, z$ on a horizon cut and a finite rectangle $\mathcal{D}$. Then

$$A_\mathcal{D} = \int_\mathcal{D} dy\, dz = \Delta y\, \Delta z, \qquad U_{\partial,\mathcal{D}} = \frac{A_\mathcal{D} ac^2}{8\pi G}. \tag{91}$$

Changing the selected patch at fixed acceleration changes $U_\partial$ and $F_L$ in proportion to area, but changes neither $L_\kappa$ nor $P_L$. No projection of a sphere is involved. This is the cleanest local demonstration that the normal pressure is not merely a Newtonian source-field formula in disguise.

For an observer who accelerates uniformly for all proper time, the Rindler horizon is a permanent causal barrier for that observer. It is nevertheless observer-relative: inertial observers in the same Minkowski spacetime do not share the partition, and a finite-duration accelerated trajectory need not possess the same eternal event horizon. The nonzero response in flat spacetime is therefore attached to the accelerated observer's boundary and clock, not to an observer-independent stress of the Minkowski vacuum.

*Normalization of the horizon energy.* The force in this section uses the same prescribed acceleration and clock as the boundary potential. For that normalization, $U_\partial = J_\eta c/L_\kappa$ and the fixed-area length response is $-F_L\, dL_\kappa$. Related quasilocal horizon-energy assignments are discussed in the antecedent literature [18]; they are not needed to introduce an additional length or energy into the present derivation. Here Rindler supplies its own exact acceleration-length interpretation, while the Schwarzschild and cosmological examples retain their stated rate prescriptions.

### B. Why the common mass–radius relation does not fix acceleration length

Surface gravity depends on geometry and the physical evolution used to normalize it. Schwarzschild's usual rate is evaluated at the horizon and normalized by the stationary clock at infinity; it is not measured at infinity. A local clock redshifts both rate and potential, without changing Schwarzschild into cosmological geometry. Evolution is a second issue: generic FRW has an evolving marginal-screen history, whereas Schwarzschild and exact de Sitter are stationary. De Sitter is their overlap, not a reason to call every FRW screen stationary.

In a spherically symmetric spacetime $ds^2 = h_{ab}dx^a dx^b + R^2 d\Omega_2^2$, the Misner–Sharp energy is

$$E_{\rm MS} = \frac{c^4 R}{2G}\left(1 - h^{ab}\partial_a R\,\partial_b R\right). \tag{92}$$

A spherical marginal surface has $h^{ab}\partial_a R\partial_b R = 0$, so $R = 2GM_{\rm MS}/c^2$, with $M_{\rm MS} = E_{\rm MS}/c^2$ [63, 64]. This is the genuine common mass–radius relation for Schwarzschild and FRW apparent horizons.

For a static spherical line element

$$ds^2 = -f(r)c^2dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2, \qquad |\kappa| = \frac{c^2}{2}|f'(R)|, \tag{93}$$

where the horizon is a simple root $f(R) = 0$, the clock is that of the displayed $t$. Standard infinity-normalized Schwarzschild and center-normalized de Sitter give, respectively,

$$\begin{aligned} f_{\rm Schw}(r) &= 1 - \frac{R_H}{r}, & f'_{\rm Schw}(R_H) &= \frac{1}{R_H}, & L_\kappa^{\rm Schw} &= 2R_H, \\ f_{\rm dS}(r) &= 1 - \frac{r^2}{R_{\rm dS}^2}, & f'_{\rm dS}(R_{\rm dS}) &= -\frac{2}{R_{\rm dS}}, & L_\kappa^{\rm dS} &= R_{\rm dS}. \end{aligned} \tag{94}$$

Both horizons are locally Rindler. Their different acceleration-length ratios are controlled by the radial metric gradients and selected normalizations; a local Rindler approximation does not fix those global data. The horizon mass–radius relation fixes $f(R) = 0$, whereas surface gravity also requires $f'(R)$. This is the geometrical reason that a shared $R = 2GM_{\rm MS}/c^2$ need not imply a shared $L_\kappa/R$. For the metric in Eq. (93), writing $f(r) = 1 - 2Gm(r)/(c^2 r)$ makes the distinction explicit:

$$\kappa_s = \frac{c^2}{2R} - \frac{Gm'(R)}{R}, \qquad m(R) = \frac{c^2 R}{2G}. \tag{95}$$

Schwarzschild has $m' = 0$; de Sitter has a volume-proportional mass profile. More general static lapse functions require their own normalization factors. Neither $L_\kappa = R$ nor $L_\kappa = 2R$ is a universal inference from the horizon's mass–radius relation alone.

### C. Schwarzschild radius work and pressure

Let $R_H = 2GM/c^2$ and normalize the stationary Killing field to unit time at infinity. Then

$$\kappa_\infty = \frac{GM}{R_H^2} = \frac{c^2}{2R_H} = \frac{c^4}{4GM}, \qquad L_\kappa = 2R_H. \tag{96}$$

With $A_H = 4\pi R_H^2$,

$$\boxed{P_L^{\rm Schw} = \frac{c^4}{32\pi G R_H^2}, \qquad F_L^{\rm Schw} = \frac{c^4}{8G}, \qquad U_\partial^{\rm Schw} = F_L L_\kappa = TS = \frac{Mc^2}{2}.} \tag{97}$$

The length-conjugate force is independent of black-hole size in four spacetime dimensions. Expressing its work in the actual areal radius $R_H$ and conventional areal volume $V_H = 4\pi R_H^3/3$ gives

$$\boxed{F_L\, dL_\kappa = F_R\, dR_H = (M\kappa_\infty)\, dR_H = P_R\, dV_H, \qquad P_R = \frac{F_R}{A_H} = 2P_L = \frac{\kappa_\infty^2}{4\pi G}.} \tag{98}$$

The radius force is $F_R = 2F_L = M\kappa_\infty = c^4/(4G)$. The coefficient doubles because $dL_\kappa = 2dR_H$; it describes the same work. More directly,

$$\frac{dL_\kappa}{L_\kappa} = \frac{dR_H}{R_H}, \qquad F_L\, dL_\kappa = U_\partial \frac{dL_\kappa}{L_\kappa} = U_\partial \frac{dR_H}{R_H}, \qquad dU_\partial = \Pi_\partial\, dA_H - P_R\, dV_H. \tag{99}$$

Both terms refer to the changing horizon sphere, without placing another boundary at $2R_H$. The areal radius is genuine geometric size, not a universal proper distance from a center or observer. Area and volume are linked: $\Pi_\partial\, dA_H = 2P_R\, dV_H$, $dU_\partial = P_R\, dV_H$, and $U_\partial = 3P_R V_H$. The potential remains $U_\partial = Mc^2/2$; the mass law is still $d(Mc^2) = T\, dS = 2dU_\partial$. No independent mass-work term is added.

### D. Exact Smarr products and their distinct responses

The Schwarzschild mass identity has the exact form

$$\boxed{Mc^2 = TS + F_L L_\kappa, \qquad TS = F_L L_\kappa = U_\partial = \frac{Mc^2}{2}.} \tag{100}$$

The two products coincide as functions of the extended sources. Their area and length derivatives nevertheless identify distinct responses:

$$TS = A\left(\frac{\partial U_\partial}{\partial A}\right)_{L_\kappa}, \qquad F_L L_\kappa = -L_\kappa \left(\frac{\partial U_\partial}{\partial L_\kappa}\right)_A. \tag{101}$$

Along the Schwarzschild family,

$$d(TS) = d(F_L L_\kappa) = dU_\partial = \frac{1}{2} d(Mc^2), \qquad T\, dS = d(Mc^2). \tag{102}$$

Thus the boundary-potential law and the standard mass law agree; no independent mass-work term or second scalar degree of freedom is added. The exact products are the result here. Their proposed kinetic/potential and stationary-virial interpretations, together with the formal free-energy relation, are examined separately in Appendix G.

### E. Spherical horizon energy and the two length scales

The general potential is area based. In the spherical marginal-horizon specialization of Eq. (92), set

$$E_{\rm sph} = E_{\rm MS} = \frac{c^4 R}{2G}, \qquad A = 4\pi R^2. \tag{103}$$

The marginal condition is essential: $c^4R/(2G)$ is not the Misner–Sharp energy of an arbitrary spherical cut. The previously established area potential becomes

$$\boxed{U_\partial = \frac{E_{\rm sph} R}{L_{\rm s}}, \qquad \frac{U_\partial}{E_{\rm sph}} = \frac{R}{L_{\rm s}} = \frac{\kappa R}{c^2}.} \tag{104}$$

Thus the ratio of the boundary potential to the geometric horizon energy is the ratio of transverse areal radius to normal acceleration length. On the positive branch $L_{\rm s} = L_\kappa$, Cai–Kim FRW has $L_\kappa = R_A$ and $U_\partial = E_A$; standard Schwarzschild has $L_\kappa = 2R_H$ and $U_\partial = Mc^2/2$. With a signed rate, Eq. (104) uses $L_{\rm s}$; magnitudes obey $|U_\partial|/E_{\rm sph} = R/L_\kappa$. The boundary potential retains its clock dependence even though $E_{\rm MS}$ is geometrically assigned.

The energy–length form unifies this interpretation with the established boost charge:

$$\boxed{U_\partial L_\mathrm{s} = cJ_\eta = E_\mathrm{sph} R.} \tag{105}$$

The first equality requires only the stated area–rate potential and Einstein boost normalization. The second uses a spherical marginal horizon. A Rindler patch therefore retains $U_\partial L_\mathrm{s} = cJ_\eta$ without any intrinsic spherical radius or enclosed Misner–Sharp energy. A formal area-equivalent radius $\sqrt{A/(4\pi)}$ labels size but does not create a curved sphere or its bulk energy.

Because $E_\mathrm{sph} \propto R$, its radius cannot be varied while holding that energy independently fixed. The complete differential is

$$\begin{aligned} dU_\partial &= \frac{R}{L_\mathrm{s}}\, dE_\mathrm{sph} + \frac{E_\mathrm{sph}}{L_\mathrm{s}}\, dR - \frac{E_\mathrm{sph} R}{L_\mathrm{s}^2}\, dL_\mathrm{s} \\ &= \frac{2U_\partial}{R}\, dR - F_L\, dL_\mathrm{s} = \Pi_\partial\, dA - F_L\, dL_\mathrm{s}. \end{aligned} \tag{106}$$

The two radius contributions together reproduce the area term; neither may be silently dropped. Section V F retains the additional changing-ratio term when $L_\mathrm{s}/R$ varies. The $ER/L_\mathrm{s}$ form is a useful spherical interpretation, not a broader replacement for $U_\partial(A, \kappa)$.

A useful distinction appears in the energy–temperature slopes. For $E = E_\mathrm{sph}$ on the positive branch, the ratio $\alpha = E/(TS)$ in the next formula equals $\alpha_R = L_\kappa/R$. For a spherical family with constant $\alpha$, $E = \alpha TS$, $S \propto R^2$, and $T \propto R^{-1}$,

$$\mathcal{C}_E \equiv \frac{dE}{dT} = -\alpha S, \qquad C_\mathrm{path} \equiv T\frac{dS}{dT} = -2S. \tag{107}$$

The second coefficient is the reversible heat-path slope when $\delta Q_\mathrm{rev} = T\, dS$ applies. Schwarzschild has $\alpha = 2$ and the mass law $dE = T\, dS$, so its energy slope is the usual heat capacity $-2S$. Cai–Kim FRW has $\alpha = 1$: its energy slope is $-S$, while the screen-law heat-path slope is $-2S$, with their difference supplied by pressure work. These are not competing definitions of a single work-free heat capacity.

At fixed area in the extended source ensemble,

$$\chi_{UT}^{(A)} \equiv \left(\frac{\partial U_\partial}{\partial T}\right)_A = S, \qquad C_A^{(Q)} \equiv T\left(\frac{\partial S}{\partial T}\right)_A = 0. \tag{108}$$

The first is an energy–temperature susceptibility. Entropy is fixed on that slice, so the variation changes the source potential through the work channel rather than transferring reversible heat.

### F. Radius force, homogeneity, and the limits of the Smarr connection

Equations (S27)–(S29) follow by substituting $A = 4\pi R^2$ and $L_\mathrm{s} = \alpha_R R$ into the parent potential:

$$U_\partial(R, \alpha_R) = \frac{c^4 R}{2G\alpha_R}, \qquad \Pi_\partial\, dA = 2F_R\, dR, \qquad F_L\, dL_\mathrm{s} = F_R\, dR + U_\partial \frac{d\alpha_R}{\alpha_R}. \tag{109}$$

The positive radius-work pressure is $P_R = F_R/A$ on the positive branch. It differs from the acceleration-length pressure by $P_R = \alpha_R P_L$. The identity

$$U_\partial = F_R R = 3P_R V = 3\alpha_R P_L V \tag{110}$$

uses the Euclidean areal-volume convention $AR = 3V$. It does not make $A\, dL_\mathrm{s}$ a general exact volume differential and does not identify $P_R$ with a matter pressure or a cosmological-constant pressure.

Under a physical rescaling with dimensionless sector data fixed, $R \mapsto \lambda R$, $A \mapsto \lambda^2 A$, $L_\mathrm{s} \mapsto \lambda L_\mathrm{s}$, and $\kappa \mapsto \kappa/\lambda$. The boundary potential has degree one:

$$U_\partial(\lambda^2 A, \lambda L_\mathrm{s}) = \lambda U_\partial(A, L_\mathrm{s}), \qquad 2\Pi_\partial A - F_L L_\mathrm{s} = U_\partial. \tag{111}$$

This is the same Euler identity as $R(\partial U_\partial/\partial R)_{\alpha_R} = U_\partial$. The radius representation displays the homogeneity underlying Smarr relations [56, 58]; the acceleration-length representation already contains it. No additional Smarr constraint follows merely from the change of variables.

For Schwarzschild, the mass first law and its integrated relation are

$$d(Mc^2) = \Pi_\partial\, dA = 2M\kappa\, dR_H, \qquad Mc^2 = 2U_\partial = 2M\kappa R_H = TS + M\kappa R_H. \tag{112}$$

In particular, $M\kappa\, dR_H = dU_\partial = \frac{1}{2}d(Mc^2)$. The two equal products $TS$ and $M\kappa R_H$ are representations of one function $U_\partial$, not independent energy reservoirs. For Cai–Kim FRW, by contrast, $\alpha_R = 1$, $U_\partial = E_A = M_A c^2$, and

$$F_R = M_A\kappa_{\rm CK} = \frac{c^4}{2G}, \qquad E_A = F_R R_A = 3P_{\rm scr}V_A, \qquad dE_A = 2F_R\, dR_A - F_R\, dR_A. \tag{113}$$

This is the analogous horizon-energy scaling identity within the Cai–Kim construction; it does not supply an asymptotic FRW mass or a stationary black-hole mass theorem.

The condition on the scale ratio is essential. For signed Hayward FRW, away from $\kappa_H^{\rm s} = 0$, the established rate relation gives $\alpha_R = 4/(3w-1)$. The second term of Eq. (109) is then $U_\partial\, d\alpha_R/\alpha_R = -3E_A\, dw/4$, exactly the equation-of-state contribution derived in Sec. VI. Only at fixed $w$ does this work reduce to $(M_A\kappa_H^{\rm s})\, dR_A$.

*a. Kerr: an exact mass–radius product with a specified radius.* For an asymptotically flat, nonextremal, uncharged Kerr black hole with the standard infinity-normalized stationary generator, let $r_+$ be the outer Boyer–Lindquist horizon coordinate and let $a_{\rm K} = J/(Mc)$. The established Kerr relations [55, 56, 58] imply

$$A = 4\pi(r_+^2 + a_{\rm K}^2) = \frac{8\pi GMr_+}{c^2}, \qquad \boxed{U_\partial = M\kappa r_+.} \tag{114}$$

Defining $\alpha_+ = L_{\rm s}/r_+$ therefore gives the exact scale-work identity

$$F_L\, dL_{\rm s} = M\kappa\, dr_+ + U_\partial \frac{d\alpha_+}{\alpha_+}, \qquad Mc^2 = 2M\kappa r_+ + 2\Omega_H J. \tag{115}$$

For $\chi = Jc/(GM^2)$ and $s = \sqrt{1-\chi^2} > 0$, $\kappa r_+/c^2 = s/2$ and $\alpha_+ = 2/s$. Hence $F_L\, dL_{\rm s} = M\kappa\, dr_+$ along fixed-dimensionless-spin scaling families. Fixed $J$ or fixed $a_{\rm K}$ is generally a different variation and retains the extra term. The standard mass first law remains $d(Mc^2) = \Pi_\partial\, dA + \Omega_H\, dJ$.

The coordinate $r_+$ is neither an areal radius nor a proper radial distance. If instead $R = \sqrt{A/(4\pi)}$ is used, the general radius product is $U_\partial = M_{\rm irr}\kappa R$, where $M_{\rm irr} = c^2R/(2G)$ is the irreducible mass, not the Kerr total mass. The spherical canonical normalization in Eq. (38) must not be transferred unchanged to $(r_+, M\kappa)$ when spin and horizon shape also vary.

*b. Charged horizons and absence of a preferred radius.* A simple counterexample to an unrestricted total-mass force claim is Reissner–Nordström. In $G = c = 1$ units, with outer areal radius $R$ and geometrized charge $Q$,

$$M = \frac{R^2+Q^2}{2R}, \quad M_R = \frac{R}{2}, \quad \kappa = \frac{1-Q^2/R^2}{2R}, \quad -\frac{A}{8\pi}\left(\frac{\partial\kappa}{\partial R}\right)_Q = \frac{1}{4}\left(1 - \frac{3Q^2}{R^2}\right). \tag{116}$$

The last coefficient is the scale-work coefficient along a fixed-$Q$ radius variation. It is generally neither $M_R\kappa$ nor $M\kappa$; the scale-ratio term accounts for the difference. At fixed $Q/R$, $\alpha_R$ is constant and the coefficient is $M_R\kappa$, while $M_R$ still differs from the total mass. Thus both the mass definition and the variation must be specified. Rindler supplies a different limit: its finite-patch boundary potential is well defined, but there is no preferred spherical radius or enclosed Misner–Sharp mass. Acceleration length remains meaningful there.

The two representations have complementary uses. Radius makes the spherical scaling and the stated Kerr Smarr product especially direct; acceleration length retains the operational Rindler normalization and the normal-rate source interpretation before a radius prescription is chosen. Neither produces an additional law merely by reparameterizing the same potential.

### G. Schwarzschild–de Sitter and clock normalization

Schwarzschild hides the normalization issue because asymptotic flatness supplies a conventional clock. For

$$f(r) = 1 - \frac{2GM}{c^2 r} - \frac{\Lambda r^2}{3} \tag{117}$$

and a static generator $\chi_\alpha^a = \alpha(\partial_t)^a$, either horizon $r_i$ has

$$|\kappa_i^{(\alpha)}| = \alpha\frac{c^2}{2r_i}|1 - \Lambda r_i^2|, \qquad P_{L,i}^{(\alpha)} = \frac{[\kappa_i^{(\alpha)}]^2}{8\pi G}. \tag{118}$$

Different admissible normalizations assign different numerical responses to the same intrinsic cut. Once a particular static observer is selected, the locally redshifted value is consistent. The absence of a preferred global clock is therefore a physical ensemble issue, not an algebraic inconsistency [48, 49].

## VI. COSMOLOGICAL APPLICATION: FRW SCREEN WORK AND NORMAL FOCUSING

### A. Compact-screen response

Consider the four-dimensional FRW metric

$$ds^2 = -c^2dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega_2^2\right), \qquad R(t,r) = a(t)r, \tag{119}$$

and let $\rho$ and $p_{\rm m}$ denote total energy density and matter pressure in the same units. The apparent-horizon radius is

$$R_A = \frac{c}{\sqrt{H^2 + kc^2/a^2}}. \tag{120}$$

The Friedmann constraint and Misner–Sharp definition give

$$E_A = \rho V_A = \frac{c^4 R_A}{2G}, \qquad V_A = \frac{4\pi}{3}R_A^3, \qquad \rho = \frac{3c^4}{8\pi G R_A^2}. \tag{121}$$

Here $V_A$ is the areal volume entering the Misner–Sharp relation. For $k \neq 0$ it differs from the proper volume of a cosmic-time spatial ball; the latter contains the radial metric factor $1/\sqrt{1-kr^2}$. Along the homogeneous family of apparent-horizon screens,

$$\boxed{P_{\rm scr} \equiv \frac{dE_A}{dV_A} = \frac{c^4}{8\pi G R_A^2} = \frac{\rho}{3}.} \tag{122}$$

This is a positive compact-screen size response, not the cosmic-fluid pressure $p_{\rm m}$ and not the conventional independent-variable derivative $-(\partial E/\partial V)_S$. Both $S_A$ and $V_A$ are functions of the single physical parameter $R_A$.

The associated force is

$$F_{\rm scr} = A_A P_{\rm scr} = \frac{c^4}{2G}, \qquad dE_A = F_{\rm scr}\, dR_A = P_{\rm scr}\, dV_A. \tag{123}$$

The force is constant in four dimensions: four times the Schwarzschild length force $F_L^{\rm Schw}$, and twice its radius force $F_R^{\rm Schw}$. At the same radius the Cai–Kim acceleration scale is twice the Schwarzschild surface gravity; the stated displacement determines which force coefficients are being compared.

The inverse pressure slope along the physical spherical family is

$$\boxed{\frac{dV_A}{dP_{\rm scr}} = -\frac{3V_A}{2P_{\rm scr}}, \qquad B_{\rm scr} \equiv -V_A\frac{dP_{\rm scr}}{dV_A} = \frac{2}{3}P_{\rm scr}.} \tag{124}$$

The factor of three relative to Eq. (D2) arises because the transverse area changes with the radius rather than being held fixed. This is a constrained-screen response slope, not a dynamical stability result. Kong's effective pressure $P_{\rm eff} \equiv -dE_A/dV_A$ is consequently $-P_{\rm scr} = -\rho/3$ in four-dimensional Einstein FRW [41]. Section IX D compares this exact overlap with the selected scale-work coefficient in Eq. (139).

### B. Friedmann system in screen-pressure variables

Define the Einstein coupling in the field-equation normalization,

$$\kappa_{\rm E} \equiv \frac{8\pi G}{c^4}. \tag{125}$$

Since $\rho = 3P_{\rm scr}$, the two Friedmann equations and the continuity equation can be written entirely in terms of the positive screen pressure $P_{\rm scr}$ and the signed matter pressure $p_{\rm m}$:

$$\frac{1}{R_A^2} = \kappa_{\rm E} P_{\rm scr}, \tag{126}$$

$$\frac{2}{c^2}\left(\dot{H} - \frac{kc^2}{a^2}\right) = -\kappa_{\rm E}(3P_{\rm scr} + p_{\rm m}), \tag{127}$$

$$\frac{2\ddot{a}}{ac^2} = -\kappa_{\rm E}(P_{\rm scr} + p_{\rm m}), \tag{128}$$

$$\dot{P_{\rm scr}} + H(3P_{\rm scr} + p_{\rm m}) = 0. \tag{129}$$

Equation (126) says that the screen pressure sets the apparent-horizon inverse-square scale. The acceleration equation shows that the universe accelerates precisely when the negative bulk pressure exceeds the positive screen pressure in magnitude, $p_{\rm m} < -P_{\rm scr}$. The combination $3P_{\rm scr} + p_{\rm m}$ governs the change of the Hubble rate and the dilution of the screen pressure. Equivalently,

$$\boxed{\frac{\dot{R}_A}{HR_A} = \frac{3P_{\rm scr} + p_{\rm m}}{2P_{\rm scr}}.} \tag{130}$$

The geometry clarifies why these combinations appear. Spherical spacetime separates as

$$ds^2 = h_{ab}dx^a dx^b + R^2(x)d\Omega_2^2, \tag{131}$$

where $h_{ab}$ is the two-dimensional Lorentzian time–radial, or orbit, geometry and the second term is the transverse symmetry sphere. On the apparent-horizon cut, the sphere has Gaussian curvature

$$\boxed{K_{S^2} = \frac{1}{R_A^2} = \kappa_{\rm E} P_{\rm scr}.} \tag{132}$$

This equality is special to the Cai–Kim realization $L_{\rm CK} = R_A$; the general boundary-pressure identity is $\kappa_{\rm E} P_L = 1/L_\kappa^2$. The scalar curvature of the normal time–radial geometry is, with the curvature convention used here,

$$\boxed{{}^{(2)}\mathcal{R}_\perp = \frac{2\ddot{a}}{ac^2} = -\kappa_{\rm E}(P_{\rm scr} + p_{\rm m}).} \tag{133}$$

The transverse two-sphere Ricci scalar is ${}^{(2)}\mathcal{R}_{S^2} = 2/R_A^2 = 2\kappa_{\rm E} P_{\rm scr}$. Their sum isolates the trace-sensitive pressure contrast,

$$\boxed{{}^{(2)}\mathcal{R}_{S^2} + {}^{(2)}\mathcal{R}_\perp = \kappa_{\rm E}(P_{\rm scr} - p_{\rm m}) = \frac{{}^{(4)}\mathcal{R}}{3}.} \tag{134}$$

Therefore $P_{\rm scr}$ fixes transverse horizon curvature, $P_{\rm scr} + p_{\rm m}$ fixes normal expansion curvature, and $P_{\rm scr} - p_{\rm m}$ fixes the trace or cross-focusing curvature.

### C. Stationary de Sitter and the evolving screen

A de Sitter static-patch horizon is both cosmological and a stationary nonexpanding Killing horizon. With a consistent clock and orientation, it therefore realizes the null and stationary corner responses directly. At fixed $\Lambda$, the cosmological term in the Einstein Lagrangian contains no metric derivatives and adds no derivative symplectic-potential term. The original cosmological area law has a gravitational Killing-identity foundation [48]; its matter Killing-energy variation is not automatically the Misner–Sharp energy of a changing horizon sphere.

For the positive equilibrium assignment $\kappa = c^2/R_A$, one has $U_\partial = E_{\rm MS} = c^4 R_A/(2G)$ and $T\,dS = 2\,dE_{\rm MS}$. Together with $P_L\,dV = dE_{\rm MS}$ this gives the screen first law. The relation does not erase the fact that adding matter at fixed vacuum density can shrink a cosmological horizon. Nor does it identify the oriented horizon charge with every energy appearing in a cosmological first law. The positive screen potential and the signed canonical orientation are separate conventions that must be matched.

A generic evolving FRW apparent-horizon worldtube is not a de Sitter Killing horizon. Vanishing expansion of one null congruence on each marginal sphere is not nonexpansion of the entire worldtube. The exact screen identities below therefore use the stated Einstein–FRW calibration, not a claim that the complete moving-boundary action has been reduced to the single stationary rate term. No new universal canonical FRW theorem is asserted.

### D. Cai–Kim equilibrium scale

A generic FRW apparent-horizon worldtube is not a non-expanding null boundary: a marginal cut does not imply a null, non-expanding tube.

A marginal sphere has vanishing expansion along one null normal. Its worldtube follows the locus of marginal spheres on successive slices, not necessarily those same light rays. In flat expanding FRW,

$$\frac{dR_\pm}{dt} = HR \pm c, \qquad R_A = \frac{c}{H}, \qquad \left.\frac{dR_-}{dt}\right|_{R_A} = 0, \qquad \dot{R}_A = -\frac{c\dot{H}}{H^2}. \tag{135}$$

These are different directional derivatives: a zero null expansion on each cut does not make the moving history nonexpanding. For flat dust FRW, $R_A = 3ct/2$ and its speed relative to the comoving frame is $\dot{R}_A - HR_A = c/2$: the tube is timelike and its area changes. A stationary de Sitter horizon is a special nonexpanding null overlap. Persistent radiation gives a null but area-growing history; marginality, nullness and nonexpansion are distinct conditions. Thus stationary Rindler, Schwarzschild, and de Sitter admit the direct restricted certification, while general CK FRW is the separate screen-state realization stated here. The null-source proposition therefore does not by itself establish a canonical theorem for this evolving screen. The following realization uses the Friedmann constraint and the separately specified instantaneous Cai–Kim calibration,

$$\kappa_{\rm CK} = \frac{c^2}{R_A}, \qquad L_{\rm CK} = R_A, \qquad T_{\rm CK} = \frac{\hbar c}{2\pi k_B R_A}. \tag{136}$$

It makes the acceleration-length pressure coincide exactly with the screen response:

$$P_R^{\rm CK} = P_L^{\rm CK} = P_{\rm scr}, \qquad T_{\rm CK} S_A = E_A, \qquad F_L^{\rm CK} = M_A \kappa_{\rm CK} = \frac{c^4}{2G}, \tag{137}$$

where $M_A = E_A/c^2$. Because $L_{\rm CK} = R_A$, the same force is conjugate to both length and radius, and $F_L^{\rm CK} dL_{\rm CK} = M_A \kappa_{\rm CK} dR_A = P_{\rm scr} dV_A$. The differential chain is

$$T_{\rm CK} dS_A = 2dE_A, \qquad -S_A dT_{\rm CK} = F_L^{\rm CK} dL_{\rm CK} = P_{\rm scr} dV_A = dE_A, \tag{138}$$

and hence

$$\boxed{dE_A = T_{\rm CK} dS_A - P_{\rm scr} dV_A.} \tag{139}$$

Here normal-scale and radial pressure coincide, whereas Schwarzschild uses $P_R = 2P_L$ for its equally exact volume work. Every quantity is fixed by the same macroscopic radius. The result is an exact reorganization of the Einstein–FRW constraint, not a new Friedmann equation and not Cai and Kim's original matter-flux identity [37].

The distinction matters. Cai and Kim used the energy supply crossing a momentarily fixed apparent horizon,

$$\delta Q_{\rm matt} = A_A(\rho + p_{\rm m}) H R_A \, dt = T_{\rm CK} dS_A, \tag{140}$$

with signs determined by orientation. Equation (139) instead resolves the simultaneous change of the constrained screen state into entropy and size legs. The flux law sorts change by carrier; the screen law sorts the same geometry by boundary coordinate.

The energy–temperature slope from Eq. (107) is

$$\frac{dE_A}{dT_{\rm CK}} = -S_A, \qquad T_{\rm CK}\frac{dS_A}{dT_{\rm CK}} = -2S_A, \qquad -P_{\rm scr}\frac{dV_A}{dT_{\rm CK}} = S_A. \tag{141}$$

The work term accounts for the difference. The second expression is the heat-path coefficient of the formal screen-state first law, not a claim of exact equilibrium thermality for arbitrary evolving FRW geometry.

### E. Full Hayward surface gravity

The Cai–Kim scale is an instantaneous equilibrium calibration. The signed Kodama–Hayward surface gravity probes the evolving normal two-geometry:

$$\kappa_H^{\rm s} = \frac{c^2}{2}\Box_h R\big|_{R_A} = -\frac{c^2}{R_A}\left(1 - \frac{\dot{R}_A}{2HR_A}\right). \tag{142}$$

The ratio form assumes $H \neq 0$. A regular expression at $H = 0$, when $R_A$ remains finite, is $\kappa_H^{\rm s} = -(R_A/2)(\dot{H} + 2H^2 + kc^2/a^2)$. It measures the transverse-null slope of the marginal condition, or normal cross-focusing. A comoving FRW observer is geodesic, so this is not the observer's proper acceleration.

Using the Friedmann and continuity equations,

$$\frac{\dot{R}_A}{2HR_A} = \frac{3P_{\rm scr} + p_{\rm m}}{4P_{\rm scr}} = \frac{3}{4}(1+w), \qquad w \equiv \frac{p_{\rm m}}{\rho}, \tag{143}$$

for arbitrary spatial curvature. Therefore the signed Hayward surface gravity admits the equivalent pressure and curvature forms

$$\boxed{\kappa_H^{\rm s} = \frac{c^2}{4R_A}(3w-1) = -\frac{c^2 R_A}{4}\kappa_{\rm E}(P_{\rm scr} - p_{\rm m}) = -\frac{c^2 R_A\,{}^{(4)}\mathcal{R}}{12}.} \tag{144}$$

It measures the trace-sensitive contrast between the positive screen pressure and the signed material pressure.

The trace combination in Eq. (144) is established horizon geometry. An optional surface-density packaging and its operational comparison tests are collected in Appendix E; they are not needed for the scale-work law. In the parent-potential identity below, $E_{\rm tr} \equiv (\rho - 3p_{\rm m})V_A$ denotes the integrated signed trace.

On a nonzero branch, the corresponding signed Hayward acceleration length is

$$\boxed{L_H^{\rm s} \equiv \frac{c^2}{\kappa_H^{\rm s}} = \frac{4R_A P_{\rm scr}}{p_{\rm m} - P_{\rm scr}} = \frac{4R_A}{3w-1}, \qquad \frac{R_A}{L_H^{\rm s}} = \frac{p_{\rm m} - P_{\rm scr}}{4P_{\rm scr}}.} \tag{145}$$

Define $F_L^H$ by first taking $-(\partial U_\partial/\partial L_{\rm s})_A$ on the independent source space and then evaluating at $(A, L_{\rm s}) = (A_A, L_H^{\rm s})$; set $P_L^H = F_L^H/A_A$. This is the prescribed Hayward-family response, with its canonical scope kept separate from the stationary null result. The quadratic response becomes

$$\boxed{P_L^H = \frac{(\kappa_H^{\rm s})^2}{8\pi G} = \frac{(P_{\rm scr} - p_{\rm m})^2}{16P_{\rm scr}} = \left(\frac{1-3w}{4}\right)^2 P_{\rm scr}.} \tag{146}$$

$P_L^H$ and $P_{\rm scr}$ answer different questions. For dust, $P_L^H = P_{\rm scr}/16$ and $|L_H^{\rm s}| = 4R_A$; for radiation the pressure contrast vanishes, so $P_L^H = 0$ and $|L_H^{\rm s}| = \infty$; at the coasting threshold $p_{\rm m} = -P_{\rm scr}$, $|L_H^{\rm s}| = 2R_A$; and for de Sitter $P_L^H = P_{\rm scr}$ and $|L_H^{\rm s}| = R_A$.

Faraoni showed that the expanding apparent horizon changes trapping and oriented-temperature character at

$$p_{\rm m} = \frac{\rho}{3} = P_{\rm scr}. \tag{147}$$

The threshold is established work; the additional reading here is that it equates matter pressure with the independently defined compact-screen response. At the threshold, $\kappa_H^{\rm s} = 0$ and $P_L^H = 0$ while $P_{\rm scr}$ remains finite. Geometrically, the positive transverse-sphere curvature and negative normal-plane curvature cancel in Eq. (134). The rate variable remains regular, but its reciprocal length chart reaches infinity [40].

A generic FRW apparent horizon is a quasi-local marginal screen rather than a cosmological event horizon. Its worldtube can be timelike, spacelike, or null, and it can be crossed as the cosmology evolves. Exact de Sitter is the special case in which the apparent and event horizons coincide.

### F. Hayward–Mukohyama–Ashworth parent potential

Hayward's work density is

$$W = -\frac{1}{2}T^{ab}h_{ab} = \frac{\rho - p_{\rm m}}{2}. \tag{148}$$

The spherical Einstein equations and the Hayward–Mukohyama–Ashworth Kodama–Noether charge give

$$\widetilde{U_\partial} \equiv \frac{A_A c^2 \kappa_H^{\rm s}}{8\pi G} = \frac{E_A}{2} - \frac{3}{2}WV_A = -\frac{1}{4}(\rho - 3p_{\rm m})V_A = -\frac{1}{4}E_{\rm tr}. \tag{149}$$

The HMA parent potential is therefore minus one quarter of the signed trace energy. Combining this identity with $\widetilde{U_\partial} = A_A c^2 \kappa_H^{\rm s}/(8\pi G)$ reproduces Eq. (E2) directly. This supplies an independent quasi-local route to the surface-density packaging: the Hayward rate is controlled by Ricci-trace energy per horizon area, while the Cai–Kim screen pressure remains controlled by total energy per screen volume.

On a branch admitting a positive Hayward temperature,

$$\boxed{T_H S_A = |\widetilde{U_\partial}| = \frac{|E_{\rm tr}|}{4}.} \tag{150}$$

The quarter in Eq. (150) and the quarter-area entropy coefficient share the same $A\kappa/(8\pi G)$ Noether normalization and a $2\pi$ conversion. They are distinct physical statements, not a microscopic derivation of one from the other.

Pulling a variation back along the physical trapping-horizon trajectory, Hayward's unified first law is

$$d_H E_A = T^{\rm s}_H d_H S_A + W\, d_H V_A. \tag{151}$$

For nonzero $\kappa^{\rm s}_H$, differentiation of Eq. (149) gives the exact allocation

$$\boxed{-F^H_L d_H L^{\rm s}_H = -\frac{1}{2} T^{\rm s}_H d_H S_A - W\, d_H V_A - \frac{3}{2} V_A d_H W.} \tag{152}$$

This shows precisely how the reciprocal-rate variation is redistributed among area, matter-work, and changing-work-density terms. The Hayward acceleration length therefore supplies a normal-scale interpretation of the established dynamical balance rather than an additional standalone first law. It is the appropriate dynamical qualification to the equilibrium Cai–Kim law [19, 38].

### G. Exact relation between the two cosmological scale-work terms

The main pressure variables are $P_{\rm scr} = \rho/3$ and the material pressure $p_{\rm m}$. For $\rho > 0$, the standard equation-of-state parameter is $w = p_{\rm m}/\rho = p_{\rm m}/(3P_{\rm scr})$. The dimensionless ratio

$$\chi \equiv \frac{\kappa^{\rm s}_H}{\kappa_{\rm CK}} = -\frac{P_{\rm scr} - p_{\rm m}}{4P_{\rm scr}} = \frac{3w-1}{4} \tag{153}$$

is a useful algebraic shorthand; it introduces no new physical source. The dynamical rate depends on the pressure contrast, whereas cosmic acceleration depends on the pressure sum. Explicitly,

$$\boxed{P^H_L = \frac{(P_{\rm scr} - p_{\rm m})^2}{16 P_{\rm scr}}, \qquad \widetilde{U_\partial} = -\frac{3}{4}(P_{\rm scr} - p_{\rm m}) V_A.} \tag{154}$$

On a branch $\chi \neq 0$, the HMA and Cai–Kim potentials obey

$$\widetilde{U_\partial} = \chi E_A, \qquad L^{\rm s}_H = \frac{R_A}{\chi}, \qquad P^H_L = \chi^2 P_{\rm scr}, \qquad F^H_L = \chi^2 F^{\rm CK}_L. \tag{155}$$

Consequently the exact scale-work transformation is

$$\boxed{F^H_L dL^{\rm s}_H = \chi P_{\rm scr} dV_A - E_A d\chi = \chi P_{\rm scr} dV_A - \frac{3}{4} E_A dw.} \tag{156}$$

In the pressure notation used for the Friedmann equations this is

$$\boxed{F^H_L dL^{\rm s}_H = \frac{p_{\rm m} - P_{\rm scr}}{4}\, dV_A - \frac{3}{4} E_A\, dw.} \tag{157}$$

The familiar $w$ is retained only to make the changing-equation-of-state contribution transparent. Equivalently, its last term is $-E_A\, d(p_{\rm m}/P_{\rm scr})/4$; no new equation-of-state variable is needed. At fixed $w$ the work reduces to $F^H_L dL^{\rm s}_H = M_A \kappa^{\rm s}_H dR_A$. On a positive-temperature branch use $f = |\chi|$, $L_H = R_A/f$, and $U^+_H = f E_A$; then $F^H_L dL_H = f P_{\rm scr} dV_A - E_A df$, with radial force $M_A|\kappa^{\rm s}_H|$ at fixed $w$.

The two energy balances are simultaneously consistent:

$$\boxed{dE_A = \Pi_{\rm CK} dA_A - P_{\rm scr} dV_A = \Pi^{\rm s}_H dA_A + W dV_A.} \tag{158}$$

Using $\Pi_{\rm CK} = R_A P_{\rm scr}$, $\Pi^{\rm s}_H = R_A(p_{\rm m} - P_{\rm scr})/4$, and $W = (3P_{\rm scr} - p_{\rm m})/2$, the same pair reads

$$\boxed{\begin{aligned} dE_A &= R_A P_{\rm scr}\, dA_A - P_{\rm scr}\, dV_A \\ &= \frac{R_A(p_{\rm m} - P_{\rm scr})}{4}\, dA_A + \frac{3P_{\rm scr} - p_{\rm m}}{2}\, dV_A. \end{aligned}} \tag{159}$$

These are balances for $E_A$, not differentials of the signed HMA potential $\widetilde{U_\partial}$. The second equality is pulled back to the physical trapping-horizon trajectory. Since $dA_A = 2dV_A/R_A$ and $2(1-\chi)P_{\rm scr} = P_{\rm scr} + W$, the different area coefficients are compensated by their different work allocations. This is an exact Einstein–FRW relation, independent of spatial curvature. It does not identify either rate with the complete source of every moving-boundary action; Appendix C gives a separate comparison that leaves this general matching problem unresolved. At $\chi = 0$, use the regular rate and potential variables instead of the singular reciprocal chart.

*a. Signed radial pressure and reconstruction of the bulk pressure.* The coefficient in Eq. (157) has a direct pressure representation. Define the signed Hayward radial coefficient by

$$\boxed{P^H_R \equiv \frac{M_A \kappa^{\rm s}_H}{A_A} = \frac{c^2 \kappa^{\rm s}_H}{8\pi G R_A} = \frac{p_{\rm m} - P_{\rm scr}}{4}, \qquad F^H_R = A_A P^H_R.} \tag{160}$$

Here $M_A = E_A/c^2 = c^2 R_A/(2G)$ is the horizon Misner–Sharp mass, not the mass equivalent of the signed Hayward boundary potential. The screen and radial response coefficients therefore reconstruct the homogeneous bulk source:

$$\boxed{\rho = 3P_{\rm scr}, \qquad p_{\rm m} = P_{\rm scr} + 4P^H_R, \qquad P^H_L = \frac{(P^H_R)^2}{P_{\rm scr}}.} \tag{161}$$

$P^H_R$ is signed and is distinct both from the nonnegative length-response density $P^H_L$ and from Hayward's work density $W$. It is the coefficient paired with $dV_A$ at fixed $w$; the full work retains $-3E_A\, dw/4$ when $w$ varies. The instantaneous reconstruction in Eq. (161) does not require constant $w$. It remains finite at $\kappa^{\rm s}_H = 0$, where the reciprocal length chart is singular.

No equation of state is needed to evaluate the two coefficients from a specified FRW geometry. For $H \neq 0$ and finite $R_A$,

$$P_{\rm scr} = \frac{c^4}{8\pi G R_A^2}, \qquad P^H_R = -P_{\rm scr}\left(1 - \frac{\dot R_A}{2HR_A}\right), \qquad p_{\rm m} = P_{\rm scr}\left(\frac{2\dot R_A}{HR_A} - 3\right). \tag{162}$$

The first two expressions use the apparent-horizon radius and the geometric Hayward rate; their identification with the bulk source uses Einstein's equations. Equivalently, $p_{\rm m} = -c^2(2\dot H + 3H^2 + kc^2/a^2)/(8\pi G)$. For an expanding spatially flat geometry, $HR_A = c$, so $R_A$ and $\dot R_A$ suffice to reconstruct the total density and pressure. For nonzero spatial curvature the displayed radius-rate formula also uses $H$. These are on-shell gravitational reconstruction identities, not an additional material equation of state or a derivation of matter microphysics. An equation of state or matter model is still needed to predict the geometry from initial data. A prescribed geometric history must in turn be compatible with that matter model. Any cosmological constant is included here in the total $\rho$ and $p_{\rm m}$.

The existing acceleration and conservation equations can correspondingly be displayed without $p_{\rm m}$:

$$\frac{\ddot a}{a} = -\frac{8\pi G}{c^2}(P_{\rm scr} + 2P^H_R), \qquad \dot{P_{\rm scr}} + 4H(P_{\rm scr} + P^H_R) = 0. \tag{163}$$

They are the same Einstein–FRW dynamics in response variables, not an independent closure relation.

*b. The same coefficients and the horizon's causal history.* In spatially flat expanding FRW, the time–radial metric gives for the apparent-horizon trajectory at fixed angles

$$\left.\frac{ds^2}{dt^2}\right|_{\mathcal H} = -c^2 + (\dot R_A - HR_A)^2 = \boxed{4c^2 \frac{P^H_R(P_{\rm scr} + P^H_R)}{P^2_{\rm scr}}}. \tag{164}$$

This is the tangent norm in cosmic-time parametrization, not a local material speed. It expresses the established causal classification of FRW apparent horizons in the two pressure variables [40]. The history is timelike for $-P_{\rm scr} < P^H_R < 0$, null at $P^H_R = 0$ (radiation equation of state) or $P^H_R = -P_{\rm scr}$ (vacuum equation of state), and spacelike outside that interval. Persistent vacuum domination gives the de Sitter stationary case; the radiation case is null but its area grows. Thus marginality of each cut, nullness of the history, and nonexpansion along that history are distinct conditions. The common pressure variables organize horizon size, normal geometry, matter pressure and causal character; they supply neither an additional horizon equation of motion nor a complete moving-boundary canonical theorem.

### H. Scope of the evolving-screen interpretation

An evolving apparent-horizon worldtube need not be null. The scalar null-boundary response does not by itself provide its complete action variation. Embedding, matter, clock, stretching and corner data must be retained when testing a cosmological rate. Appendix C finds a mismatch for one specified timelike ensemble; this excludes that identification in that ensemble, not every possible prescription. The additional surface-density comparisons are kept with their tests in Appendix E. The exact equilibrium and Hayward state identities above do not depend on interpreting the packaged density as a material membrane or a literal shell.

## VII. INTEGRABILITY AND FINITE DYNAMICAL BALANCE

### A. When the scale leg is and is not a state energy

On the extended boundary source space, define the scale one-form

$$\alpha_L \equiv -F_L\,\boldsymbol{\delta} L_{\rm s} = \frac{Ac^2}{8\pi G}\,\boldsymbol{\delta}\kappa. \tag{165}$$

Its phase-space exterior derivative is

$$\boxed{\mathbf{d}_\Gamma \alpha_L = \frac{c^2}{8\pi G}\boldsymbol{\delta}A\wedge\boldsymbol{\delta}\kappa = -P_L\boldsymbol{\delta}A\wedge\boldsymbol{\delta}L_{\rm s}.} \tag{166}$$

Therefore there is no universal state function $H_L(A, L_{\rm s})$ satisfying $\boldsymbol{\delta}H_L = \alpha_L$ when area and scale vary independently. Here "independently" means virtual source variations in the extended boundary ensemble, not a claim that every physical history explores an arbitrary two-dimensional path. The principal Schwarzschild and Cai–Kim families are one-dimensional and link $A$ to $L_\kappa$; Rindler also supplies an operational setting in which patch area and observer acceleration can be controlled separately. The nonzero curl does not disqualify the term as work–ordinary $P\,dV$ is generally nonexact–and does not qualify the pressure law on a physical one-parameter family. It rules out only a universally separate stored scale energy on the unrestricted source space.

The complete parent variation remains exact because

$$\mathbf{d}_\Gamma(\Pi_\partial\boldsymbol{\delta}A) = -\mathbf{d}_\Gamma\alpha_L, \qquad \boldsymbol{\delta}U_\partial = \Pi_\partial\boldsymbol{\delta}A + \alpha_L. \tag{167}$$

On a one-dimensional physical family, or any leaf $L_{\rm s} = L_{\rm s}(A)$, the pullback of the two-form vanishes and the scale leg is integrable along that leaf. Ashtekar, Fairhurst, and Krishnan obtained the equivalent condition $\boldsymbol{\delta}\kappa\wedge\boldsymbol{\delta}A = 0$ for isolated-horizon Hamiltonian evolution [42].

For spherical Hayward horizons, using

$$\kappa_H^{\rm s} = \frac{c^2}{2R} - \frac{4\pi GR}{c^2}W \tag{168}$$

in Eq. (166) gives the exact reduced obstruction

$$\boxed{\mathbf{d}_\Gamma\alpha_L = -\boldsymbol{\delta}V\wedge\boldsymbol{\delta}W.} \tag{169}$$

The isolated scale leg is therefore integrable on restricted leaves such as fixed $W$, $W = W(V)$, or a single history, but not for generic independent variations of screen size and equation of state.

### B. Finite constant-force law for dynamical black holes

Ashtekar, Paraizo, and Shu proved an exact nonperturbative charge–flux law for quasi-local dynamical black-hole horizons that can be arbitrarily far from equilibrium [43]. Their preferred equilibrium projection assigns each marginal cut the Kerr surface gravity $\bar\kappa(A, J)$. In their geometric units the phase-space charge is $Q_{\rm PS} = \bar\kappa A/(8\pi G)$ and obeys

$$\Delta Q_{\rm PS} = \mathfrak{F}^{(\xi)}_{\rm gws} + \mathfrak{F}^{(\xi)}_{\rm matt}. \tag{170}$$

Restoring acceleration units and defining

$$L_{\rm APS}=\frac{c^2}{\bar\kappa},\qquad U_{\rm APS}=\frac{Ac^2\bar\kappa}{8\pi G},\qquad F_L^{\rm APS}=\frac{A\bar\kappa^2}{8\pi G},\tag{171}$$

gives the exact acceleration-length representation

$$\boxed{\Delta U_{\rm APS}=\int_{S_1}^{S_2}\left(\Pi_\partial\, dA-F_L^{\rm APS}dL_{\rm APS}\right)=E^{(\xi)}_{\rm gws}+E^{(\xi)}_{\rm matt}.}\tag{172}$$

APS supply the second equality; the first is the present source representation. Matter and gravitational fluxes do not separately map onto the area and scale legs.

In the nonrotating sector the equilibrium projection is Schwarzschild:

$$\bar\kappa=\frac{c^2}{2R},\qquad L_{\rm APS}=2R,\qquad U_{\rm APS}=\frac{c^4R}{4G},\qquad F_L^{\rm APS}=\frac{c^4}{8G}.\tag{173}$$

The force is exactly constant. Along this projected family $A=\pi L_{\rm APS}^2$ and $\Pi_\partial\, dA=2F_L^{\rm APS}dL_{\rm APS}$, so the full differential is $dU_{\rm APS}=\Pi_\partial\, dA-F_L^{\rm APS}dL_{\rm APS}=F_L^{\rm APS}dL_{\rm APS}$. The positive total change retains both legs of the differential. Consequently the finite physical-process law becomes

$$\boxed{E^{(\xi)}_{\rm gws}+E^{(\xi)}_{\rm matt}=\Delta U_{\rm APS}=F_L^{\rm APS}\,\Delta L_{\rm APS}.}\tag{174}$$

This is a finite charge–flux law expressed in the projected acceleration length. It does not identify that length with a proper material displacement during the dynamical process.

APS normalize the nonrotating horizon mass as $E_{\rm DH}=2U_{\rm APS}$. Since $U_{\rm APS}=T_{\rm proj}S=F_L^{\rm APS}L_{\rm APS}$ on every projected cut, with $T_{\rm proj}=\hbar\bar\kappa/(2\pi k_Bc)$,

$$\boxed{\Delta(T_{\rm proj}S)=\Delta(F_LL)=\Delta U_{\rm APS}=\frac{1}{2}\Delta E_{\rm DH}.}\tag{175}$$

The two projected Euler products have the same finite change because both equal $U_{\rm APS}$. The nontrivial additional result is that APS relate this charge change to finite matter and gravitational fluxes. This does not establish independent energy reservoirs, assign either product to one flux carrier, or imply a KMS state during arbitrary evolution. $T_{\rm proj}$ is the temperature parameter of the preferred equilibrium projection.

For rotating cuts, $A$, $J$, $\bar\kappa$, and $\bar\Omega$ are linked by the Kerr projection and the rotational channel must be retained. The constant-force simplification is specific to the nonrotating four-dimensional sector.

## VIII. STATIONARY WALD-ENTROPY AND HIGHER-CURVATURE EXTENSION

The acceleration-length idea has a clean stationary extension, but two notions must be separated: the universal $S_W\,\delta T$ channel and the specific membrane pressure of a higher-curvature theory.

Chakraborty and Padmanabhan establish complementary thermal variations with sector- and normalization-dependent factors [44]. In their v2, static Eqs. (69)–(70) use $mU$; the restricted generic-null Eqs. (84)–(85) have $1/m$ for the unrescaled $m$th-order surface term. These statements do not supply an unqualified higher-curvature null-boundary theorem. The following stationary lift is the algebraic response of the specified thermodynamic $S_W dT$ channel. Holding Wald entropy fixed and using $T=\hbar c/(2\pi k_BL_\kappa)$ gives

$$S_W\,dT=-F_L^{(W)}dL_\kappa,\qquad \boxed{F_L^{(W)}=\frac{TS_W}{L_\kappa},\qquad P^{(W)}_{L,SdT}=\frac{F_L^{(W)}}{A}=\frac{\hbar c}{2\pi k_BL_\kappa^2}s_W,}\tag{176}$$

where $s_W=S_W/A$. For Einstein gravity, $s_W=k_B/(4\ell_{\rm P}^2)$ and Eq. (176) reduces exactly to $\kappa^2/(8\pi G)$.

The membrane pressure in a pure $m$th-order Lanczos–Lovelock theory has an additional theory- and dimension-dependent factor. Kolekar and Kothawala found, for the isotropic horizon sector with maximally symmetric spatial cuts used in their derivation,

$$\frac{\Pi_\partial^{(m)}A}{T}=\frac{D-2m}{D-2}S_W^{(m)}.\tag{177}$$

At fixed Wald entropy density and couplings, its acceleration-length susceptibility is therefore

$$\boxed{P^{(m)}_{L,\mathrm{mem}} = -\left(\frac{\partial \Pi^{(m)}_{\partial}}{\partial L_\kappa}\right)_{s_W} = \frac{D-2m}{D-2}\frac{\hbar c}{2\pi k_B L_\kappa^2} s^{(m)}_W.} \tag{178}$$

For a sum of Lovelock densities, the corresponding stationary membrane response is the sum of the weighted entropy densities. Einstein gravity is the $m = 1$ case and the factor equals one. In the critical dimension $D = 2m$, the pure Lovelock term is topological: it can contribute a global Wald entropy while its local membrane pressure factor vanishes. This is a separate critical-dimension limiting observation, not an extension of the membrane source's proved non-topological range [45].

The distinction between Eqs. (176) and (178) is important. The first is the response of the thermodynamic $S_W \delta T$ channel. The second is the derivative of the actual Lanczos–Lovelock membrane pressure. They coincide in Einstein gravity but not universally in higher-curvature theories. Moreover, fixed area need not mean fixed Wald entropy. Along a physical family for which $S_W$ changes with $L_\kappa$,

$$-\frac{d(TS_W)}{dL_\kappa} = \frac{TS_W}{L_\kappa} - T\frac{dS_W}{dL_\kappa}, \tag{179}$$

so the second term belongs to the entropy/area sector rather than the fixed-entropy scale response.

For stationary $f(R)$ gravity with $f_R$ effectively constant over the source variation, $s_W = f_R k_B/(4\ell_{\mathrm{P}}^2)$ and the Wald-channel result becomes

$$P^{f(R)}_{L,SdT} = f_R \frac{\kappa^2}{8\pi G}. \tag{180}$$

Generic variations of $f_R$ and generic dynamical higher-curvature horizons add derivative and non-Newtonian membrane terms. A complete covariant null-source proof in those theories remains separate work. The result established here is the stationary Wald-channel lift and its precise relation to the known Lovelock membrane equation of state.

## IX. GENERALITY, ANTECEDENTS, AND INTRINSIC SIGNIFICANCE

### A. Common potential, canonical scope, and physical realizations

Horizon boundary mechanics names the mechanics of the specified Einstein horizon boundary potential. It does not assert a complete theory of every boundary response. The same potential and normal-rate prescription give the same mechanical normal form; canonical support, physical displacement, and geometric work retain their boundary conditions.

The covariant realization has three levels:

1. The generic covariant Einstein null scalar source is $\mu = \widehat{\kappa} + \theta/2$.
2. On a clock-fixed non-expanding null sector, $\mu = \widehat{\kappa}$ and the branchwise coordinate change to $L_{\mathrm{s}} = 1/\widehat{\kappa}$ gives the exact Hamilton–Jacobi response $P_L = \kappa^2/(8\pi G)$.
3. In other sectors, an independently established normal-rate prescription and parent $A\kappa$ potential can support the same algebra, but not automatically the same reduced canonical theorem.

Squaring removes the orientation sign, not the clock dependence. Extremal or trace-boundary points with $\kappa = 0$ must be described in the regular parent rate rather than the reciprocal chart.

The pullback and normal-balance results sharpen the physical hierarchy. The generic source $\mu$ is the broadest null variable. The linear surface pressure/tension $\Pi_\partial$ appears directly in tangential and normal screen equations. The quadratic $P_L$ is the normal susceptibility of that stress with respect to the mechanically privileged acceleration length. It is not more general than $\mu$, but it is more directly analogous to ordinary three-dimensional pressure because it is force per area conjugate to the selected normal length. Its conversion to geometric volume work requires the corresponding displacement dictionary.

In $D$ spacetime dimensions, if a spherical sector has $A \propto R^{D-2}$ and $L_\kappa \propto R$, then

$$F_L = AP_L \propto R^{D-4}. \tag{181}$$

The scale-independent force is therefore special to four spacetime dimensions. This dimensional fact is independent of the numerical normalization selected in a particular sector.

### B. Compact theorem and restricted canonical support

The compact acceleration-length derivation and the full covariant analysis carry different kinds of information. Within the inherited standard horizon-mechanics sector, the established stress first fixes $\kappa_{\mathrm{E}}\Pi_\partial = \kappa/c^2 = 1/L_{\mathrm{s}}$. The potential and its short differential chain

$$U_\partial(A, L_{\mathrm{s}}) = \frac{Ac^4}{8\pi G L_{\mathrm{s}}}, \qquad dU_\partial = \frac{U_\partial}{A}\, dA - \frac{U_\partial}{L_{\mathrm{s}}}\, dL_{\mathrm{s}} \tag{182}$$

is already a self-contained conditional theorem. Its derivatives give $\Pi_\partial$, $F_L = (U_\partial/c^2)\kappa$, $P_L = F_L/A$, and the pressure-work form; Rindler, Schwarzschild, and Cai–Kim FRW then follow after their standard sector dictionaries are supplied. The clock, generator normalization, relational cut, and non-expanding or equilibrium assumptions are inherited from the same horizon mechanics that defines the parent potential, membrane stress, and Hawking temperature.

The full construction identifies the action source behind the compact result. In the stated null-boundary sector it derives the selected response from the mixed null symplectic potential and directly from the EH current, reproduces the same coefficient through the complementary area–boost corner action, and records the clock, gauge, sign, zero-surface-gravity, integrability, prescription, and dynamical limits. The direct calculation and mathematical counterexamples are in Appendix I. Some extensions follow from the compact normal form, while others–notably the off-family integrability obstruction, finite APS support, and higher-curvature distinctions–supply additional information rather than merely expanding the chain rule.

This separation is useful conceptually. The structural insight is the identification of acceleration length as the normal mechanical coordinate of the $A\kappa$ boundary potential. The formal analysis shows exactly how that insight is realized by general relativity, where it is inherited from standard horizon mechanics, and where it cannot be extended without further input.

### C. Relation to prior work

The ingredients on which the construction rests are established: area–boost conjugacy; local near-horizon $A/\ell$ energy; the $A\kappa$ Kodama–Noether charge; the $T\delta S + S\delta T$ surface-term split; the null scalar pair and mixed boundary conditions; the Damour–Navier–Stokes membrane pressure; Freidel and Yokokura's Young–Laplace screen equation; Hayward's unified first law; the Cai–Kim apparent-horizon construction; and the APS finite dynamical charge–flux theorem [11–13, 16–20, 24, 25, 28, 37, 38, 43].

The closest antecedent to the central coefficient is the action-derived $A\,\delta\kappa$ response, not only the known potential value $TS$. Table I distinguishes that inherited structure from the acceleration-length interpretation and its consequences. Equation numbers in the table refer to the specified source versions; all factors of $c, G$ are restored in the manuscript's own equations. Section IX D separately compares the explicit FRW and Schwarzschild pressure formulas of Kong and Ong.

The novelty claim concerns the length-conjugate force and normal-scale pressure obtained from this established boundary structure, together with their specified action response and geometric work interpretation. Recognizing the normalized work length turns the rate response into $-F_L\, dL_{\mathrm{s}}$ and identifies $P_L = F_L/A$. The $A\kappa$ potential and the $S\,dT$ channel are the inputs; they are not being relabeled as discoveries. The construction develops the following consequences of those inputs; this list does not claim that every special-case formula is new:

1. the length-conjugate force and normal-pressure identification $P_L = \kappa^2/(8\pi G)$, $F_L = AP_L$, with action and corner realizations in the stated restricted sectors;

2. the direct pressure-work form $-P_L \vartheta_V^{(\delta)}$;

3. the signed pullback relation $D_A\Pi_\partial = -P_L D_A L_{\mathrm{s}}$ (equivalently $D_A|\Pi_\partial| = -P_L D_A L_\kappa$) and the normal balance $\Delta p - P_L L_{\mathrm{s}} H = 0$;

4. the mechanical–thermal bridge laws $P_L \propto L_\kappa^{-2}$ and $T \propto L_\kappa^{-1}$, together with the exact inverse-energy/entropy relations $k_B T/F_L = \bar{\lambda}_\partial/(2\pi) = L_\kappa/N_S$ and $S/k_B = 2\pi L_\kappa/\bar{\lambda}_\partial$;

5. the Schwarzschild Euler–Smarr identities and radius-work realization;

6. the positive Cai–Kim FRW screen pressure and compact first law;

7. the integrability criterion; supporting response functions are in Appendix D; and

TABLE I. Closest prior results and the precise use made of them here. This targeted source comparison is not an exhaustive historical-priority search.

| Primary source and locator | Established result adopted | Present coordinate/response consequence |
|---|---|---|
| Parattu–Majhi–Padmanabhan, 1303.1535v2, Eqs. (111)–(113) [20] | Static horizon-preserving variations give $\kappa\delta A/(8\pi G)$ and $A\delta\kappa/(8\pi G)$, thermally $T\delta S$ and $S\delta T$. | The prescribed $L_{\rm s} = c^2/\kappa$ expresses the existing rate coefficient as $-F_L\delta L_{\rm s}$, with normal pressure $P_L = F_L/A$. |
| Hopfmüller–Freidel, 1611.03096v2, Eqs. (4.27), (5.3)–(5.5); 1802.06135v1, Eqs. (1.3)–(1.4) [11, 12] | Null area/rate structure; in four dimensions $\mu = \widehat{\kappa} + \theta/2$ and the scalar rate term is $\epsilon_{\mathcal{N}}\delta\mu$. | Selected rate polarization and tangent nonexpansion give the length response. The scalar pair is inherited; the coordinate change adds no degree of freedom. |
| Odak–Rignon-Bret–Speziale, 2309.03854v3, Eqs. (3.26)–(3.28), (4.6), (4.8) [13] | Mixed null polarization, fixed $\theta + 2k$, and additional corner-area condition. | The explicit action in Appendix I identifies the prescription; nonzero source response compares neighboring fixed-source problems. |
| Hayward–Mukohyama–Ashworth, gr-qc/9810006v1, Eqs. (7), (9), (12); Cai–Kim, hep-th/0501055v1, Eqs. (2.12)–(2.17) [19, 37] | Spherical Einstein work/energy relations, Kodama charge $A\kappa/(8\pi G)$, and instantaneous equilibrium FRW temperature assignment. | The prescribed screen has $L_{\rm s} = R_A$, $U_{\rm eq} = E_{\rm MS}$ and $P_{\rm scr} = \rho/3$. The signed Hayward coefficient is paired with $dV_A$ at fixed $w$; its reconstruction of $p_{\rm m}$ and the full $dw$ term are distinguished after Eq. (161). |
| Ashtekar–Paraizo–Shu, 2604.00170v3, Eqs. (5.7)–(5.10) [43] | Preferred equilibrium projection, generator, phase-space charge and finite charge–flux theorem. | The nonrotating four-dimensional specialization admits the constant-force length representation in Eq. (174). |
| Kolekar–Kothawala, 1111.1242v3, Eq. (31); Chakraborty–Padmanabhan, 1408.4791v2, Eqs. (69)–(70), (84)–(85) [44, 45] | Wald thermal channels and Lovelock membrane stresses, with theory-, order- and normalization-dependent factors. | The stationary $S_W\delta T$ length response and the weighted membrane derivative remain separate in Sec. VIII. |

8. the nonrotating APS finite constant-force flux law and the stationary Wald-entropy lift.

The contribution belongs to the same scientific archetype as the membrane reformulation of horizon dynamics, Iyer–Wald's Noether-charge organization, Jacobson's equation-of-state derivation, Hayward's unified first law, Padmanabhan's horizon thermodynamic identity, Cai–Kim's FRW construction, and the Frodden–Ghosh–Perez local first law: established gravity can support new identifications of thermodynamic or mechanical quantities and relationships without changing its equations [10, 18, 21, 37–39]. The comparison concerns contribution type: explanatory and mechanical content need not change the underlying field equations. Historical reach and priority are separate questions. In particular, Padmanabhan's $p_r\,dV$ is genuine matter-pressure work in a horizon field-equation identity. Here the coefficient is the response of the specified boundary potential; for vacuum Schwarzschild that scale-work term is nonzero even though the matter radial pressure vanishes. Frodden–Ghosh–Perez already supply local inverse-distance energy, with observer distance held fixed in their area law. The present comparison admits the prescribed scale variation and retains each sector's clock. These are differences of potential, variation, and response, not dismissals of the established macroscopic interpretations.

### D. Equation-level comparison with Kong and Ong

The following four-dimensional comparisons use geometrized units $G = c = 1$; the products $TS$ retain the usual Hawking and area-entropy normalizations. They identify published overlaps, not historical priority over the author's stated disclosure record.

*a. Kong: the same FRW energy and total volume slope.* Kong's arXiv:2507.17789v5, Eqs. (3.6)–(3.10) (HTML Eqs. (19)–(23)), specializes to [41]

$$\begin{aligned} E = E_A = \frac{R_A}{2}, \qquad & V = V_A = \frac{4\pi R_A^3}{3}, \\ P_{\rm eff} = -\frac{dE_A}{dV_A} = -\frac{1}{8\pi R_A^2} = -\frac{\rho}{3} = -P_{\rm scr}, \quad & E_A = -3P_{\rm eff}V_A = 3P_{\rm scr}V_A. \end{aligned} \tag{183}$$

Thus the magnitude and energy–volume identity coincide exactly. Kong combines the dependent entropy and volume terms into a single effective slope. In the present equilibrium-screen assignment, the same total derivative is resolved as

$$\underbrace{dE_A = -P_{\rm eff} dV_A}_{\text{total family slope}} = \underbrace{T_{\rm CK} dS_A - P_{\rm scr} dV_A}_{\text{area and scale responses}}, \qquad T_{\rm CK} dS_A = 2P_{\rm scr} dV_A. \tag{184}$$

The distinction is the response being represented, not a new scalar coefficient obtained by changing a sign. The selected work leg follows from $F_L = -(\partial U_\partial/\partial L_{\rm s})_A$ before restricting $L_{\rm s} = R_A$ and $A = 4\pi R_A^2$; on that family its magnitude equals $dE_A$. This FRW specialization remains an exact prescribed-screen identity, not a general moving-boundary canonical theorem.

*b. Ong: half-energy, radial work, and the volume measure.* Ong's arXiv:2309.04110v3, Sec. IV, Eqs. (10)–(16), uses [57]

$$\begin{aligned} V_{\rm shell} &= 4\pi R_H^2 \epsilon, & P_{\rm shell} &= \frac{1}{32\pi R_H \epsilon}, \\ P_{\rm shell} dV_{\rm shell} &= \frac{1}{4}\, dR_H \quad (d\epsilon = 0), & E &= \frac{M}{2} = TS = 2TS - 2P_{\rm shell} V_{\rm shell}. \end{aligned} \tag{185}$$

The unnumbered spherical-volume paragraph immediately before Sec. V also explicitly gives $P = 1/(16\pi R_H^2)$, exactly the present $P_R$. Ong does not adopt that alternative: his stated Smarr expression would give $E = 2M/3$. Here the spherical Euler identity instead reads

$$\begin{aligned} V_H &= \frac{4\pi R_H^3}{3}, & P_R &= \frac{1}{16\pi R_H^2} = 2P_L, \\ F_L\, dL_\kappa &= P_R\, dV_H = \frac{1}{4}\, dR_H, & U_\partial &= \frac{M}{2} = TS = 2TS - 3P_R V_H. \end{aligned} \tag{186}$$

The half-energy, spherical pressure coefficient, and radial work magnitude therefore have explicit published counterparts. The added construction identifies $U_\partial$ as the specified boundary potential and derives its scale response using the selected action and clock, with $L_\kappa = 2R_H$. It requires no shell thickness or maximum-force premise. The coefficients two and three in the Euler expressions reflect the different volume measures and scaling prescriptions. Area and spherical volume remain linked; the standard mass law $dM = T\, dS = 2dU_\partial$ is retained. The shared force value alone establishes no universal force bound.

To our knowledge, the cited treatments do not develop the same normalized acceleration-length force and pressure construction with the source prescription and geometric work dictionary given here. This is the specific novelty claim to assess, rather than the existence of the parent differential or of radial work in general. The formulas' dependence on established inputs does not imply that the proposed response identification had already been developed; conversely, an absence of a known equivalent is not a proof of exhaustive historical priority.

### E. Questions of interpretation and explanatory significance

*a. Does a known differential eliminate the contribution?* No. The established rate response and its thermal interpretation specify the structure from which the new force and pressure are identified. The claim is the displacement-adapted response, its action role, and the common geometric work description. Mathematical equivalence to the rate form is necessary for consistency, not evidence that the same mechanical identification was already developed. It also means that the change of variable alone supplies no new field equation or physical degree of freedom.

*b. Why this work coordinate?* The length-selection argument in Sec. II specifies its class: lengths built only from $c$ and the prescribed nonzero $\kappa$, normalized by Rindler distance. It does not exclude equally valid coordinates with transformed coefficients. The useful diagnostic is the full spherical conversion,

$$F_L\, dL_{\rm s} = \frac{U_\partial}{R}\, dR + U_\partial\, d\ln|\alpha_R|, \qquad \alpha_R = L_{\rm s}/R.$$

Schwarzschild and the equilibrium FRW screen fix $\alpha_R$ to two and one. A changing ratio requires the second contribution. Its signed Hayward-FRW realization is the equation-of-state term already derived in Sec. VI. This tests the adequacy of radius-only bookkeeping; it is not independent empirical confirmation of the parent potential. Rindler retains a planar cut and a normal separation, not a spherical radius.

*c. Do different canonical and cosmological inputs defeat unity?* The common result is a boundary-potential state law. The selected nonexpanding-null source theorem and the Einstein–Misner–Sharp plus equilibrium-rate FRW realization establish different statements about that law. Their coexistence is consistent, but the former cannot be silently applied to an arbitrary moving FRW tube. Likewise, changing from the equilibrium to the Hayward rate changes the potential and the allocation of work. The full dynamical source-matching problem remains the distinct problem identified in the moving-screen appendix.

*d. What is fixed when force is defined?* The physical evolution and clock, action and boundary prescription, cuts, and remaining controlled sources are specified. The length response compares nearby admitted boundary-value problems. A constant physical clock rescaling changes the normalized response in the known way; changing a source-dependent boundary term can shift the response itself. Neither operation is the same as a coordinate reexpression within one prescription. The claim is a definite response in that prescription, not an invariant pressure assigned to an unnormalized cut.

*e. What can explanation establish without new predictions?* A formulation can identify new useful response quantities, relate previously separate energy accounts, and clarify when one work coordinate is adequate, while remaining equivalent to established gravity. Those are the aims here. The identities $F_L = U_\partial/L_\mathrm{s}$, $F_L = M_\partial\kappa$, and $P_L = F_L/A$ share their inputs and must not be counted as independent tests. The comparative value is the common prescription and geometric work interpretation, not a collection of algebraically equivalent confirmations. Claims of empirical deviations would require additional dynamics and evidence not supplied here.

*f. Which extensions are established, and which are interpretation?* The stationary Wald-channel lift in Sec. VIII is an explicit conditional extension; the complete moving-boundary matching problem remains separate. Exact source susceptibilities, additional cosmological comparisons, and an imported inflationary spectrum are distinguished in Appendices D–F. Appendix G preserves the proposed kinetic/potential and quantum-phase readings without asserting a separate derivation of those interpretations. Appendix H records conditional scale alignments, not quantum-gravity predictions. These appendices are not premises of the main mechanical result.

The author dates the first public positive FRW screen-work result to May 10, 2025, with the corresponding record: `https://x.com/JTT329/status/1921341959009902709`. The acceleration-length formulation is dated June 2026 in the author's chronology; this manuscript contains subsequent refinements. These dates identify the stated milestones, not a proof that every later result was in the initial disclosure. Kong's distinct effective-pressure preprint was first submitted on July 23, 2025 [41]; its negative volume-slope convention is compared at equation level in Sec. IX D. The inflationary relations in Appendix F are separate consequences obtained by importing the established tensor spectrum, whose inverse-entropy scaling is not claimed as new.

## X. CONCLUSION

Recognizing acceleration length as a geometrically normalized work coordinate of the established Einstein boundary potential identifies its conjugate boundary force and normal-scale pressure:

$$\boxed{U_\partial = \frac{Ac^4}{8\pi G L_\mathrm{s}}, \qquad dU_\partial = \Pi_\partial\, dA - F_L\, dL_\mathrm{s}, \qquad F_L = \frac{U_\partial}{L_\mathrm{s}}, \quad P_L = \frac{F_L}{A} = \frac{\kappa^2}{8\pi G}.} \tag{187}$$

The contribution is this response identification and its geometric work interpretation, rather than the established potential or its product-rule differential. The null scalar source reduces to surface gravity on the prescribed nonexpanding sector. Its mixed action response and stationary corner representation provide the action-level support, with the clock, cuts, boundary terms, and permitted variations retained. The force is the coefficient of the allowed length-source variation; this canonical role does not require $U_\partial$ to be the entire Hamiltonian. The common length is selected within the stated class of geometric lengths by Rindler distance normalization; other coordinates remain valid with transformed conjugate coefficients.

For Rindler, $L_\kappa$ is a normal observer–horizon separation and the planar patch area is independent. In Schwarzschild, $L_\kappa = 2R_H$, so

$$F_L\, dL_\kappa = M\kappa\, dR_H = P_R\, dV_H, \qquad P_R = 2P_L. \tag{188}$$

Both area and radius contributions describe the same changing horizon sphere. The potential is $U_\partial = Mc^2/2$; its law leaves $d(Mc^2) = T\, dS$ unchanged and gives the mechanical Smarr form $Mc^2 = TS + F_L L_\kappa$. The products $TS$ and $F_L L_\kappa$ coincide, while their conjugate derivatives distinguish area and scale responses.

The separately calibrated equilibrium FRW screen has $L_\kappa = R_A$ and $U_\partial = E_A$, yielding

$$P_\mathrm{scr} = \frac{\rho}{3}, \qquad dE_A = T_\mathrm{CK}\, dS_A - P_\mathrm{scr}\, dV_A. \tag{189}$$

This state law and the pressure representation of the Friedmann system retain the Einstein–Misner–Sharp constraint and the chosen equilibrium rate. They are not a replacement for the matter-crossing flux law or an unrestricted moving-boundary canonical theorem. The Hayward realization has its own signed radial coefficient $P_R^H$, distinct from $P_L^H$ and $W$. The on-shell reconstruction $\rho = 3P_{\rm scr}$, $p_{\rm m} = P_{\rm scr} + 4P_R^H$ connects the two responses to the bulk source and, in flat expanding FRW, its horizon's causal history. It is neither a new material equation of state nor a moving-boundary canonical theorem. For a general spherical scale ratio $\alpha_R = L_{\rm s}/R$, the complete conversion is $F_L\, dL_{\rm s} = (U_\partial/R)\, dR + U_\partial\, d\ln|\alpha_R|$; the pure radius-work form therefore requires the specified fixed-ratio family.

The membrane pullback and matched normal-balance equation connect the new response to established force-balance structure. Integrability, finite dynamical-charge comparisons, and the stationary Wald-entropy extension retain their stated hypotheses. Thermal identification is additional to the mechanics and gives $U_\partial = TS$ and the exact inverse-energy/entropy relations. The supporting appendices separate source derivatives and applications with extra inputs from optional kinetic/potential, phase, and Planck-scale interpretations. Appendix I gives the explicit action calculation and counterexamples that establish the need for its premises. The prior-work comparison locates the claimed contribution in the mechanical interpretation of established boundary structure, without asserting exhaustive historical priority. None of the optional extensions is needed to establish the central result: one boundary potential, surface stress paired with area change, and force paired with a geometrically identified horizon-scale change.

## Appendix A: Signed and positive acceleration length

The canonical derivation uses the signed source $L_{\rm s} = c^2/\kappa$. On a fixed-sign branch define

$$\sigma \equiv \operatorname{sgn}(\kappa), \qquad L_\kappa = \frac{c^2}{|\kappa|} = |L_{\rm s}|, \qquad L_{\rm s} = \sigma L_\kappa. \tag{A1}$$

The signed boundary potential and surface stress are then

$$U_\partial = \sigma \frac{Ac^4}{8\pi G L_\kappa}, \qquad \Pi_\partial = P_L L_{\rm s} = \sigma P_L L_\kappa, \qquad |\Pi_\partial| = P_L L_\kappa. \tag{A2}$$

Their spatial derivatives obey

$$D_A\Pi_\partial = -P_L D_A L_{\rm s} = -\sigma P_L D_A L_\kappa, \qquad D_A|\Pi_\partial| = -P_L D_A L_\kappa, \tag{A3}$$

and the source work form becomes

$$\Theta_L^{(0)} = -\sigma \int d\tau \int_S \epsilon_S\, P_L \delta L_\kappa. \tag{A4}$$

$P_L$ and all positive magnitudes are orientation independent, while the signed stress and work one-form carry $\sigma$. This is why the surface stress changes sign while the pressure does not: orientation reversal changes both $\Pi_\partial$ and $L_{\rm s}$, leaving the signed ratio $\Pi_\partial/L_{\rm s} = P_L$ invariant. With the positive length one instead writes $|\Pi_\partial|/L_\kappa = P_L$. Neither reciprocal coordinate covers $\kappa = 0$.

## Appendix B: FRW rate identities

Einstein's equations in the conventions of Eq. (119) are

$$H^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3c^2}\rho, \tag{B1}$$

$$\dot H - \frac{kc^2}{a^2} = -\frac{4\pi G}{c^2}(\rho + p_{\rm m}), \tag{B2}$$

$$\dot\rho + 3H(\rho + p_{\rm m}) = 0. \tag{B3}$$

Differentiating Eq. (120) gives

$$\dot R_A = \frac{4\pi G}{c^4} H R_A^3 (\rho + p_{\rm m}), \tag{B4}$$

and hence

$$\frac{\dot{R}_A}{2HR_A} = \frac{2\pi G R_A^2}{c^4}(\rho + p_{\rm m}) = \frac{3}{4}(1+w), \tag{B5}$$

proving Eq. (143) for arbitrary $k$ where $H \neq 0$. At $H = 0$ the regular non-ratio expressions in the main text apply.

The spherical surface-gravity identity

$$\kappa_H^{\rm s} = \frac{GE_A}{c^2 R_A^2} - \frac{4\pi G R_A}{c^2} W \tag{B6}$$

immediately gives Eq. (149); differentiating it and using Eq. (151) gives Eq. (152).

## Appendix C: Timelike spherical boundary comparison and unresolved horizon matching

This appendix preserves a separate boundary-action investigation; it is not part of the principal horizon-pressure claim. The calculation below establishes a local Hamilton–Jacobi identity for one specified spherical timelike mixed ensemble, with the stated matter, clock, and endpoint conditions. It does *not* establish that this ensemble's response is the full canonical response to the original Hayward or Cai–Kim acceleration length on an independently moving horizon. An explicit FRW comparison gives different rates. The general matching problem remains open. Throughout this appendix set $c = 1$.

### 1. Action, matter, clock, and endpoint conditions

Let $\mathcal{B}$ be a smooth timelike spherical worldtube with induced metric

$$\gamma_{ij}dy^i dy^j = -N_B^2(t)dt^2 + R_B^2(t)d\Omega^2, \qquad A_B = 4\pi R_B^2. \tag{C1}$$

Its embedding may vary. Take Einstein gravity with minimally coupled scalar matter, the unreferenced Gibbons–Hawking–York term on $\mathcal{B}$, the corresponding cap terms, and the standard oriented joint terms:

$$I_D = \frac{1}{16\pi G}\int_{\mathcal{M}} \mathcal{R}\epsilon + I_\phi + \frac{1}{8\pi G}\int_{\mathcal{B}} K\epsilon_{\mathcal{B}} + I_{\rm caps} + I_{\rm joints}. \tag{C2}$$

For example, $I_\phi = -\int[(\nabla\phi)^2/2 + V(\phi)]\epsilon$. There is no reference subtraction or additional source-dependent counterterm. Fix the pulled-back scalar boundary values and take local variations compactly supported away from the caps and their joints. The joint terms are included; their endpoint variations then vanish. If endpoints are varied, their area–boost contributions must be restored. No assertion of vanishing matter energy flux is made.

For embedding displacement $\chi^a$, write $\delta_X$ for variation of the pullback to a fixed reference boundary. Thus $\delta_X\phi = X^*(\delta\phi + \mathcal{L}_\chi\phi) = 0$ is the matter condition, not $\delta\phi = 0$ at a fixed spacetime point. Normal motion contributes $2\chi_\perp K_{ij}$ to $\delta_X\gamma_{ij}$ in addition to the metric and tangential-reparameterization terms. It is included in the metric variation, not discarded by identifying the boundaries.

Use outward normal $n^a$, $K_{ij} = \gamma_i{}^a\gamma_j{}^b\nabla_a n_b$, and the unreferenced boundary stress

$$\tau^{ij} = \frac{K\gamma^{ij} - K^{ij}}{8\pi G}. \tag{C3}$$

The established on-shell action variation, including nonorthogonal boundary geometry, gives [59, 60]

$$\delta I_D\big|_{\mathcal{B}} = \frac{1}{2}\int_{\mathcal{B}} \epsilon_{\mathcal{B}}\tau^{ij}\delta_X\gamma_{ij} = \int dt\,[-E_B\delta_X N_B + N_B p_B \delta_X A_B]\,. \tag{C4}$$

Here $E_B = A_B\tau_{ij}u^i u^j$ and $p_B$ is the isotropic surface stress, with force-per-length units after restoring $c$. Matter source variations vanish by the stated Dirichlet condition. Spherical symmetry removes tangential momentum and shear responses. The moving region may equivalently be pulled back to a fixed reference region; diffeomorphism covariance then includes the embedding variation in the pulled-back fields. The bulk field equations and constraints are used, so there is no additional independent on-shell shape term omitted from Eq. (C4) under these conditions.

### 2. The area-mixed action and its exact source response

Choose the boundary Legendre transform and define a rescaled surface-stress source

$$I_M = I_D - \int dt\, N_B A_B p_B, \qquad \zeta_B = -8\pi G N_B p_B. \tag{C5}$$

This is a spherical area-mixed polarization with a separate clock source; it is not an identification with the full three-dimensional York conformal ensemble. Before introducing $\zeta_B$, the variation is

$$\delta I_M\big|_{\mathcal{B}} = \int dt[-E_B \delta_X N_B - A_B \delta_X (N_B p_B)]. \tag{C6}$$

The first term comes from varying the boundary lapse, whose conjugate is the Brown–York energy in this source convention. The second is the surface-stress source obtained by exchanging area for its conjugate. Thus the displayed coefficient $A_B/(8\pi G)$ in the next equation partly reflects the *definition* $\zeta_B = -8\pi G N_B p_B$. It is not, by itself, evidence that $\zeta_B$ is a horizon surface gravity. Brown–York boundary variation is established; this particular spherical Legendre transform and source normalization are choices made here for the comparison. Direct substitution gives

$$\boxed{\delta I_M\big|_{\mathcal{B}} = \int dt \left[-E_B \delta_X N_B + \frac{A_B}{8\pi G}\delta_X \zeta_B\right].} \tag{C7}$$

On a branch $\zeta_B \neq 0$, let $L_B = 1/\zeta_B$. Then

$$\boxed{\delta I_M\big|_{\mathcal{B}} = \int dt\, [-E_B \delta_X N_B - F_B \delta_X L_B], \quad F_B = \frac{A_B \zeta_B^2}{8\pi G}.} \tag{C8}$$

With the boundary clock $N_B$ fixed, this is the complete local Hamilton–Jacobi boundary variation for the specified spherical ensemble and matter/endpoint conditions, wherever the on-shell action is defined. The response is $-(\delta I_{M,\mathrm{os}}/\delta L_B)_{N_B,\phi} = F_B$ and its force per area is $\mathcal{P}_B = \zeta_B^2/(8\pi G)$. As in the null theorem, source variations compare nearby boundary-value problems; conservative variations within one problem fix the selected sources. This variational identity does not assert existence or uniqueness of every initial-boundary value problem.

The surface stress is determined geometrically. Put

$$a_n = n_a u^b \nabla_b u^a, \qquad b_B = \frac{n^a \nabla_a R_B}{R_B}. \tag{C9}$$

The spherical extrinsic-curvature eigenvalues give

$$E_B = -\frac{A_B b_B}{4\pi G}, \qquad p_B = \frac{a_n + b_B}{8\pi G}, \qquad \boxed{\zeta_B = -N_B(a_n + b_B).} \tag{C10}$$

The factor $8\pi G$ in $\zeta_B$ removes the Einstein normalization, $N_B$ converts the local stress contribution to the selected time normalization, and the minus sign records the chosen mixed-source orientation. This normalization makes the comparison transparent but does not by itself identify a horizon rate. The geometric eigenvalues provide the independent content: $a_n$ is the normal acceleration of the timelike observers, and $b_B$ measures normal transverse-size change per radius. The broader source is therefore a clock-weighted extrinsic boundary-stress rate, not surface gravity by definition.

The rescaled source therefore includes both normal acceleration and cut extrinsic curvature. It is not a priori a horizon surface gravity. Writing $\kappa_B = c^2 \zeta_B$ would restore acceleration units and give $\mathcal{P}_B = \kappa_B^2/(8\pi G)$ in the corresponding clock convention, but would not supply the missing geometric identification.

In the language of the main paper, one can also form the auxiliary state function $U_B = A_B \zeta_B/(8\pi G)$ and $\Pi_B = \zeta_B/(8\pi G) = -N_B p_B$. Its differential is

$$dU_B = \Pi_B\, dA_B - F_B\, dL_B. \tag{C11}$$

This exact algebraic identity is distinct from the action variation: $U_B$ is not generally the Brown–York energy $E_B$, the Misner–Sharp energy, or the horizon potential $U_\partial$ for a prescribed $\kappa_H^{\mathrm{s}}$ or $\kappa_{\mathrm{CK}}$. Likewise, $L_B = 1/\zeta_B$ is a reciprocal boundary-stress coordinate, not an independently measured proper displacement. It agrees with the original $c^2/\kappa$ only where the chosen geometry, orientation, and clock establish that $\zeta_B = \kappa/c^2$. No replacement of the main paper's definition of acceleration length is proposed.

### 3. Explicit flat-FRW moving-screen check

In cosmic-time/areal-radius coordinates, use

$$ds^2 = -(1 - H^2R^2)dt^2 - 2HR\,dt\,dR + dR^2 + R^2d\Omega^2. \tag{C12}$$

For an arbitrary timelike embedding $R = R_B(t)$, define its velocity relative to the comoving flow and its lapse by

$$v = \dot{R}_B - HR_B, \qquad N_B = \sqrt{1 - v^2}, \qquad \gamma_v = (1 - v^2)^{-1/2}. \tag{C13}$$

The comoving orthonormal frame gives

$$a_n = \gamma_v^3\dot{v} + \gamma_v Hv, \qquad b_B = \frac{\gamma_v(1 + HR_Bv)}{R_B}. \tag{C14}$$

Substitution in Eq. (C10) yields the exact source

$$\zeta_B = -\left[\frac{\dot{v}}{1 - v^2} + 2Hv + \frac{1}{R_B}\right]. \tag{C15}$$

On the timelike part of the actual apparent horizon, $R_A = 1/H$, put $q = \dot{R}_A = -\dot{H}/H^2$. Then $v = q - 1$ and $0 < q < 2$, so

$$\boxed{\zeta_B = \frac{1 - 2q}{R_A} - \frac{\dot{q}}{q(2 - q)}, \quad \kappa_{\rm CK} = \frac{1}{R_A}, \quad \kappa_H^{\rm s} = \frac{q/2 - 1}{R_A}.} \tag{C16}$$

For an Einstein–FRW background $q = 3(1 + w)/2$. A homogeneous scalar can realize a background with constant $w = 0$; then

$$\zeta_B = -\frac{2}{R_A}, \qquad \kappa_{\rm CK} = \frac{1}{R_A}, \qquad \kappa_H^{\rm s} = -\frac{1}{4R_A}. \tag{C17}$$

For these dust observers, the separate contributions are $N_Ba_n = 1/(2R_A)$ and $N_Bb_B = 3/(2R_A)$. Removing the $b_B$ contribution from the bracket leaves $-N_Ba_n = -1/(2R_A)$, neither desired rate. Holding that contribution fixed must be justified in an admitted variation; its contribution cannot be dropped merely to rename the remaining acceleration surface gravity. This is a result for the selected ensemble, alongside the positive restricted charge comparisons below, not a universal obstruction. An orientation reversal changes the sign of $\zeta_B$, not its magnitude. This is an explicit demonstration that the natural rate of this specified timelike mixed action differs from both cosmological rates. It is not a disproof of the separate Cai–Kim screen identity or the HMA Noether potential, nor a no-go theorem for every possible boundary ensemble. The endpoints $q = 0, 2$ are null and require a limit or a null action; the timelike expression must not be used there without qualification.

The inclusion of normal motion can be checked directly at the action level. For a homogeneous minimally coupled scalar on an Einstein–FRW solution, $p = -(2\dot{H} + 3H^2)/(8\pi G)$. Integrating the full on-shell Einstein-plus-scalar bulk action inside $R_B(t)$ and adding the timelike GHY term gives, up to the fixed cap and joint terms,

$$I_D = \frac{1}{G}\int dt\left\{\frac{R_B^3}{6}(\dot{H} + 3H^2) + \frac{R_B^2}{2}\left[\frac{\dot{v}}{1 - v^2} + 3Hv + \frac{2}{R_B}\right]\right\}. \tag{C18}$$

Vary $R_B(t)$ at fixed background and with compact time support; the pulled-back scalar values are fixed because the scalar is homogeneous. The resulting Euler–Lagrange shape variation agrees identically with Eq. (C4), including $\delta_X v = \delta\dot{R}_B - H\delta R_B$ and the induced lapse variation. This checks the embedding contribution using the complete bulk and boundary action, not only a change of thermodynamic coordinates.

For a stationary metric $ds^2 = -f(r)dt^2 + dr^2/f(r) + r^2d\Omega^2$, the inner-boundary orientation gives $\zeta_B = f'/2 + f/r$. At a nonextremal horizon its limit is $f'(R_H)/2$, the Killing surface gravity in the selected time normalization. This establishes the stationary Schwarzschild limit in that orientation and clock normalization; it does not establish generic cosmological matching. For a de Sitter static patch, the outer-boundary orientation instead gives $\zeta_B = -f'/2 - f/r$, tending to $+1/R_H$ when $f = 1 - r^2/R_H^2$. This equals the positive Cai–Kim magnitude and the magnitude of the signed Hayward rate. The latter is $-1/R_H$ in the cosmological convention, so comparison of signed rates also requires matching orientation. These are limits of stationary stretched boundaries, not evaluations of Eq. (C16) at its null endpoint $q = 0$.

### 4. Relation to the FRW gravitational surface charges

A complementary calculation keeps the Einstein–Hilbert charge prescription instead of changing the boundary ensemble. In flat FRW, with $K = \partial_t$, fixed areal cut, and variations $\delta H(t)$, the Einstein potential and Noether charge give

$$\int_S Q_K = -\frac{R^3}{4G}(\dot{H} + 2H^2), \tag{C19}$$

$$\int_S \iota_K \theta_{\rm EH} = -\frac{R^3}{4G}(\delta\dot{H} + 8H\delta H), \tag{C20}$$

$$\int_S (\delta Q_K - \iota_K \theta_{\rm EH}) = \frac{HR^3}{G}\delta H = \delta E_{\rm MS}\big|_R. \tag{C21}$$

For $X = \partial_t + z(t)\partial_R$, with the displayed radial extension, the adjusted charge is

$$\int_S (\delta Q_X - Q_{\delta X} - \iota_X \theta_{\rm EH}) = \frac{R^2}{2G}(2HR - z)\delta H. \tag{C22}$$

Evaluating on $R_A = 1/H$, setting $z = \dot{R}_A$, and restricting to $\delta H = -\delta R_A/R_A^2$ gives $\kappa_H^{\rm s}\delta A/(8\pi G)$. These are restricted gravitational surface-charge identities, not replacements for the full timelike action variation above. They identify precisely where the Misner–Sharp energy and Hayward area coefficient enter.

Spatial curvature also matters for this charge prescription. With $b = k/a^2$, $D = 1 - bR^2 > 0$, $\dot{b} = -2Hb$, and $K = \sqrt{D}\,\partial_t$, the same fixed-cut gravitational calculation gives

$$\int_S (\delta Q_K - Q_{\delta K} - \iota_K \theta_{\rm EH}) = \delta\left[\frac{R^3(H^2 + b)}{2G}\right]_R + \frac{R^5(b - \dot{H})}{8GD}\delta b. \tag{C23}$$

No cancellation of the extra term by matter or other boundary choices is claimed. The all-curvature HMA and constrained-screen identities in the main text are unaffected.

This result also clarifies the relation to the stretched-Carrollian potential of Freidel and Jai-akson, whose scalar sector retains clock, area, and stretching responses even in spherical symmetry [61]. Here the clock response is displayed in Eq. (C7); the timelike metric packages the stretching and motion into $N_B$, $a_n$, and $b_B$. Their more general source space has not been identified with a single Hayward-rate variation. Establishing that identification would require the corresponding action, rigging, and source restrictions explicitly.

The established result of this appendix is the displayed local variational identity for the chosen ensemble, together with the explicit action and surface-charge checks. The unresolved step is a canonical calculation for the intended physical horizon prescription whose complete response can be identified with the original $\kappa_H^{\rm s}$ or $\kappa_{\rm CK}$ while all other sources and embedding contributions are treated consistently. The unequal rates above prevent that identification for this ensemble on a generic FRW screen; they are not a no-go theorem for other ensembles. No new universal acceleration length or horizon-pressure law is claimed here.

## Appendix D: Supporting response functions

**Status: exact source-space derivatives.** These calculations follow from the same potential under the fixed-variable conditions stated below. They are supporting consequences, not separate empirical confirmations or a material stability theory.

### 1. Compliance, modulus, and source-space Hessian

The inverse pressure derivative defines a source compliance. At fixed transverse area,

$$\mathcal{V}_\partial = AL_\kappa, \qquad P_L = \frac{c^4}{8\pi G L_\kappa^2} = \frac{c^4 A^2}{8\pi G \mathcal{V}_\partial^2}. \tag{D1}$$

Therefore

$$\boxed{\left(\frac{\partial \mathcal{V}_\partial}{\partial P_L}\right)_A = -\frac{\mathcal{V}_\partial}{2P_L}.} \tag{D2}$$

The associated scale compressibility and bulk modulus are

$$\chi_L^{(A)} \equiv -\frac{1}{\mathcal{V}_\partial}\left(\frac{\partial \mathcal{V}_\partial}{\partial P_L}\right)_A = \frac{1}{2P_L}, \qquad B_L^{(A)} \equiv -\mathcal{V}_\partial\left(\frac{\partial P_L}{\partial \mathcal{V}_\partial}\right)_A = 2P_L. \tag{D3}$$

The positive modulus means that increasing the normal scale lowers its conjugate pressure at fixed area. It is a source susceptibility; a restoring motion or stability criterion additionally requires a load, an equation of motion, and boundary conditions.

Using Eq. (72),

$$P_L(T) = \frac{\pi c^2 k_B^2}{2G\hbar^2}T^2, \qquad \frac{dP_L}{dT} = \frac{2P_L}{T}, \tag{D4}$$

and at fixed area

$$\mathcal{V}_\partial T = \frac{A\hbar c}{2\pi k_B}, \qquad \alpha_L^{(A)} \equiv \frac{1}{\mathcal{V}_\partial}\left(\frac{\partial \mathcal{V}_\partial}{\partial T}\right)_A = -\frac{1}{T}. \tag{D5}$$

These are response functions of the acceleration-length representation.

The Hessian of $U_\partial = \mathcal{T}_G A/L_\kappa$ on the extended source space is

$$\frac{\partial^2 U_\partial}{\partial(A, L_\kappa)^2} = \begin{pmatrix} 0 & -P_L \\ -P_L & 2F_L/L_\kappa \end{pmatrix}, \qquad \det = -P_L^2 < 0. \tag{D6}$$

The potential is a saddle if $A$ and $L_\kappa$ are treated as independent sources. This is not by itself a dynamical instability because the variables define a boundary ensemble, not an unconstrained material state space. The positive fixed-area modulus in Eq. (D3) is the relevant local scale response.

## Appendix E: Additional cosmological comparisons and their limits

**Status: derived identities with separately tested interpretations.** These formulas re-express the Einstein–FRW equations using the screen variables. They are not new cosmological dynamics. In particular, an energy-per-area packaging does not establish a physical surface layer; the negative operational tests below are part of the result, not optional qualifications that can be dropped when quoting it.

### 1. Ricci-trace surface density

The dependence of the FRW Kodama–Hayward rate and temperature on $\rho - 3p_{\rm m}$, together with the radiation threshold $p_{\rm m} = \rho/3$, is established in the apparent-horizon literature [40]. The additional step here is to package that trace combination as a signed surface density. Define

$$E_{\rm tr} \equiv (\rho - 3p_{\rm m})V_A, \qquad M_{\rm tr} \equiv \frac{E_{\rm tr}}{c^2}, \qquad \Sigma_{\rm tr} \equiv \frac{M_{\rm tr}}{A_A}. \tag{E1}$$

$E_{\rm tr}$ is the signed Ricci-trace combination multiplied by the areal volume $V_A$, equal to $-V_A T^a{}_a$ in the $(-+++)$ signature. For curved FRW slices this is not generally a proper-volume integral. It is not generally a conserved material energy or a positive rest mass. Using $V_A/A_A = R_A/3$ and $\kappa_{\rm CK} = c^2/R_A$ gives

$$\boxed{\Sigma_{\rm tr} = \frac{(\rho - 3p_{\rm m})R_A}{3c^2} = \frac{P_{\rm scr} - p_{\rm m}}{\kappa_{\rm CK}}, \qquad \kappa_H^{\rm s} = -2\pi G\Sigma_{\rm tr} = -\frac{GM_{\rm tr}}{2R_A^2}.} \tag{E2}$$

The exact statement is that the signed Hayward cross-focusing rate is determined by the bulk Ricci-trace mass equivalent per horizon area. Where a Hayward temperature is meaningful,

$$\boxed{k_B T_H = \frac{\hbar|\kappa_H^{\rm s}|}{2\pi c} = \frac{\hbar G}{c}|\Sigma_{\rm tr}|.} \tag{E3}$$

Faraoni's temperature formula already contains this result in the unpackaged form $R_A(\rho - 3p_{\rm m})$; Eq. (E2) is therefore a compact boundary-density representation rather than a new dynamical equation.

A complementary active surface density is

$$\boxed{\Sigma_{\rm act} \equiv \frac{(\rho + 3p_{\rm m})V_A}{c^2 A_A} = \frac{P_{\rm scr} + p_{\rm m}}{\kappa_{\rm CK}}, \qquad \frac{\ddot{a}}{a} R_A = -4\pi G \Sigma_{\rm act}.} \tag{E4}$$

The pair separates two familiar projections of Einstein's equation: $\rho + 3p_{\rm m}$ controls timelike relative acceleration, whereas $\rho - 3p_{\rm m}$ controls the Ricci trace and FRW Hayward cross-focusing. The third combination $\rho + p_{\rm m} = 3P_{\rm scr} + p_{\rm m}$ controls null energy flux and the evolution of $R_A$.

The broader four-dimensional spherical Hayward identity in Eq. (168) can itself be written as

$$\boxed{\kappa_H^{\rm s} = 4\pi G(\Sigma_E - \Sigma_W), \qquad \Sigma_E \equiv \frac{E}{Ac^2}, \qquad \Sigma_W \equiv \frac{RW}{c^2}.} \tag{E5}$$

FRW homogeneity gives $\Sigma_E - \Sigma_W = -\Sigma_{\rm tr}/2$. The trace form is not the general spherical law: in Vaidya null dust $W = 0$ and the matter trace vanishes while $\kappa_H = GE/(c^2R^2)$ remains nonzero; in inhomogeneous dust the first term contains the average Misner–Sharp density while the work term contains local matter data. Homogeneity is what turns the general surface-density balance into the simple Ricci-trace density.

The coefficient $2\pi G$ is the same numerical conversion that gives the one-sided Newtonian field of an ideal plane, or the mean self-field of a thin spherical shell. No distributional shell is present in smooth FRW, so this is a mass-sheet form rather than a demonstrated mass sheet.

### 2. Quintessence in screen-pressure variables

Because $P_{\rm scr} = \rho/3$, accelerated expansion is the pressure inequality $p_{\rm m} < -P_{\rm scr}$. If a canonical scalar field dominates, let $K_\phi$ and $V_\phi$ denote its kinetic and potential energy densities, so $\rho_\phi = K_\phi + V_\phi$ and $p_\phi = K_\phi - V_\phi$; the acceleration condition becomes $V_\phi > 2K_\phi$. The canonical accelerating-quintessence range $-1 < w_\phi < -1/3$ is equivalently

$$\boxed{-3P_{\rm scr} < p_\phi < -P_{\rm scr}, \qquad R_A < |L_H^{\rm s}| < 2R_A, \qquad \frac{1}{4}P_{\rm scr} < P_L^H < P_{\rm scr}.} \tag{E6}$$

The cosmological-constant limit has $|L_H^{\rm s}| = R_A$ and $P_L^H = P_{\rm scr}$, while the coasting limit has $|L_H^{\rm s}| = 2R_A$ and $P_L^H = P_{\rm scr}/4$.

### 3. Operational audit of the trace-surface-density interpretation

*a. Published membrane and Brown–York tests.* A direct FLRW membrane construction by Wang uses a generator for which $g_H = H$ and, in units $c = 1$, obtains at the apparent horizon

$$\rho_H = 0, \qquad \widetilde{p}_H = \frac{H}{8\pi G}. \tag{E7}$$

The result is an oblique, generator-dependent membrane, and its surface energy density is not $\Sigma_{\rm tr}$ [50]. The Brown–York analysis of Oltean, Bazrafshan Moghaddam, and Epp likewise gives for rigid FLRW spheres a quasilocal energy density proportional to $\gamma^{-1}$, which tends to zero as the rigid worldtube approaches the apparent horizon [51]. If the flat-space vacuum term is subtracted, the limiting mass per area is instead $c^2/(4\pi G R_A)$, independent of $w$, whereas

$$\Sigma_{\rm tr} = (1 - 3w)\frac{c^2}{8\pi G R_A}. \tag{E8}$$

The two agree only accidentally at $w = -1/3$. These established quasilocal constructions therefore do not identify $\Sigma_{\rm tr}$ as the physical membrane energy density.

*b. Junction test.* A literal infinitesimal shell requires a discontinuity of first metric derivatives and a delta-function surface stress. For stationary null horizons, the corresponding surface stress is controlled by jumps such as $[\kappa]$ [52]. A smooth FRW apparent horizon has no such jump. Equation (E2) therefore describes a smooth quasi-local rate, not an Israel or Barrabès–Israel shell.

*c. Normal-frame boost test.* The Hayward rate is also not the time derivative of the naive rapidity of the apparent-horizon worldtube. In spatially flat dust FRW, $a(t) \propto t^{2/3}$ and $R_A = 3ct/2$. The horizon moves relative to the comoving flow with constant speed $\dot{R}_A - HR_A = c/2$, so that rapidity is constant, while $\kappa^{\mathrm{s}}_H = -c^2/(4R_A) \neq 0$. Any corner interpretation must therefore use the Kodama-adapted normal-bundle connection and moving-cut terms, not the elementary worldtube velocity.

*d. Dimensional test.* For $n$ spatial dimensions, the Einstein–FRW equations give

$$\frac{\dot{R}_A}{2HR_A} = \frac{n}{4}(1+w), \qquad \frac{\kappa^{\mathrm{s}}_H R_A}{c^2} = \frac{n(1+w)-4}{4}. \tag{E9}$$

This is proportional to the stress trace $\rho - np$ only for $n = 3$. The simple $\rho - 3p$, $-E_{\mathrm{tr}}/4$, and $-2\pi G\Sigma_{\mathrm{tr}}$ relations are therefore specifically four-dimensional.

*e. What a broader canonical theorem would mean.* Two extensions are different: a null hypersurface with nonzero expansion and shear, and a marginal-horizon worldtube that can be timelike, spacelike, or null. Equation (31) controls one scalar part of the first problem, not the complete second problem. A full result for a specified cosmological rate would derive the on-shell variation with the actual embedding, matter, clock, stretching, and corner data, then identify the coefficient of the original intended $L_{\mathrm{CK}}$ or $L^{\mathrm{s}}_H$ while retaining the other terms. The explicit mismatch in Appendix C excludes that identification for its chosen ensemble, not for every possible ensemble. The missing matching step is not completed by renaming its natural source as surface gravity.

*f. Scope of the moving-boundary comparison.* A Kodama-adapted Carrollian polarization remains a useful test of the HMA stress $\Pi_H = c^2\kappa^{\mathrm{s}}_H/(8\pi G) = -c^2\Sigma_{\mathrm{tr}}/4$ [53, 61]. It must retain clock and stretching responses, not only the pressure–area pair. Appendix C examines an explicit timelike area-mixed action. Its rescaled surface stress differs from $\kappa^{\mathrm{s}}_H$ on a generic evolving screen, so that calculation does not complete a Hayward or Cai–Kim canonical extension. Tracking, peeling, and Kodama–Hayward rates must be compared within specified prescriptions [54].

## Appendix F: An application with additional quantum input: inflationary tensors

**Status: application of an imported quantum spectrum.** The boundary state law is not a derivation of primordial fluctuations. This appendix preserves the pressure/entropy representation of the standard spectrum, with its quantum-state and cosmological assumptions kept explicit. It is not an input to the force–pressure construction.

The equilibrium screen pressure also gives a mechanical representation of a standard quantum observable. Consider spatially flat, quasi-de Sitter inflation in four-dimensional Einstein gravity, with constant $G$, the usual Bunch–Davies vacuum, and linear tensor perturbations without additional tensor sources. The tensor action and its quantization are extra inputs; they do not follow from the boundary state law. We use the instantaneous Cai–Kim assignment $L_\kappa = R_A = c/H$ and $\kappa_{\mathrm{CK}} = cH$, not the full dynamical Hayward rate. Denote a comoving wave number by $q$ to distinguish it from the spatial-curvature parameter $k$. A star denotes evaluation at its Hubble-scale exit, $cq = a_*H_*$.

*a. Amplitude, pressure, and entropy.* The dimensionless primordial tensor power $\Delta^2_t(q)$ is the variance per logarithmic wave-number interval, summed over the two polarizations. It is not a pressure, radiated power in watts, or the energy density of gravitational waves. The conventional leading slow-roll result is [70]

$$\Delta^2_t(q) \simeq \frac{16\hbar G H_*^2}{\pi c^5}. \tag{F1}$$

The same background has

$$P_{\mathrm{scr}} = \frac{c^2H^2}{8\pi G}, \qquad N_S \equiv \frac{S_A}{k_B} = \frac{A_A}{4\ell^2_{\mathrm{P}}} = \frac{\pi c^5}{\hbar G H^2}, \qquad P_{\mathrm{Pl}} \equiv \frac{c^7}{\hbar G^2}. \tag{F2}$$

Here $P_{\mathrm{Pl}}$ is a defined Planck pressure, not an assumed upper bound. The pressure–entropy product and tensor representation are

$$P_{\mathrm{scr}}N_S = \frac{P_{\mathrm{Pl}}}{8}, \qquad \boxed{\Delta^2_t(q) \simeq 128\frac{P_{\mathrm{scr},*}}{P_{\mathrm{Pl}}} = \frac{16}{N_{S,*}} = \frac{16\ell^2_{\mathrm{P}}}{\pi L^2_{\kappa,*}}.} \tag{F3}$$

The first equality is an exact identity of the stated background assignments; the spectrum retains the leading slow-roll accuracy of Eq. (F1). Larger screen pressure corresponds to a smaller inflationary horizon, lower horizon entropy, and greater tensor power. Fluctuation *amplitude*, rather than power, scales as $N_S^{-1/2}$. The entropy is that of the inflationary horizon at the relevant exit time, not today's cosmological horizon.

The inverse-entropy scaling is established in the literature, not a new claim of this construction. Brustein and Medved explicitly write $P_T(H) \sim 1/S_{\rm dS}$ in their Eq. (70) and compare with the standard inflationary result [71]. Their $P_T$ denotes tensor *power*, and their entropy is dimensionless in their unit convention; that notation is unrelated to $P_L$ or $P_{\rm scr}$. This citation records prior recognition of the scaling, without adopting their microscopic model or its numerical normalization. Entropy-based inflationary analyses also predate it [72]. The present form relates the standard spectrum to the mechanically defined force-per-area response rather than introducing a new tensor spectrum.

*b. Tilt as pressure evolution.* Let $\mathcal{N} = \ln(a/a_{\rm ref})$ be increasing elapsed e-folds and $\epsilon_H = -\dot{H}/H^2$. The background definitions give exactly

$$\frac{d\ln P_{\rm scr}}{d\mathcal{N}} = -2\epsilon_H, \qquad \frac{d\ln N_S}{d\mathcal{N}} = 2\epsilon_H. \tag{F4}$$

At first order in slow roll, the tensor tilt becomes

$$n_t \equiv \frac{d\ln\Delta_t^2}{d\ln q} \simeq -2\epsilon_{H,*} = \left.\frac{d\ln P_{\rm scr}}{d\mathcal{N}}\right|_* = -\left.\frac{d\ln N_S}{d\mathcal{N}}\right|_*. \tag{F5}$$

The standard $n_t \simeq -2\epsilon_H$ is used here [70]. Along exit times, $d\ln q = (1-\epsilon_H)d\mathcal{N}$; identifying these logarithmic increments is a first-order slow-roll step, not an exact all-orders statement about the spectrum. Tensor amplitude probes the pressure scale, while tilt probes its fractional evolution. A slowly falling screen pressure produces a red tensor tilt, or equivalently a growing horizon entropy.

*c. Scalar normalization and inference.* At a reference wave number $q_\star$, define $A_s = \Delta^2_{\mathcal{R}}(q_\star)$ and $r = \Delta_t^2(q_\star)/A_s$. Then

$$P_{\rm scr,*} \simeq \frac{rA_s}{128}P_{\rm Pl}, \qquad N_{S,*} \simeq \frac{16}{rA_s}. \tag{F6}$$

These are alternative parameterizations of the same inferred inflationary scale; they are not independent measurements of a local pressure. Under the additional assumptions of canonical single-field slow-roll inflation with the standard scalar vacuum spectrum, $r \simeq 16\epsilon_{H,*}$ [70], and therefore

$$r \simeq -8\left.\frac{d\ln P_{\rm scr}}{d\mathcal{N}}\right|_* = 8\left.\frac{d\ln N_S}{d\mathcal{N}}\right|_*, \qquad r \simeq -8n_t. \tag{F7}$$

The scalar assumption is needed for this last relation, not for the pressure substitution in the tensor amplitude.

The pressure in these expressions is the homogeneous horizon response, not a proposed tensor-noise source. No molecular interpretation of $N_S$ is required. A boundary derivation of the fluctuations would still have to reproduce the tensor two-point function, normalization, and state dependence from specified dynamical and quantum inputs. Changing the tensor action, initial state, entropy law, or horizon-rate prescription can alter the correspondence. Equations (F3)–(F7) are consequently a consistency target and a mechanical interpretation of established inflationary predictions, not a derivation of those predictions from the classical boundary potential alone.

## Appendix G: Interpretive connections and open physical hypotheses

**How to read this appendix.** The exact algebraic relations below are distinguished from proposed physical readings. In particular, the kinetic/potential split is an interesting interpretation of the equal Schwarzschild products, not a separately derived decomposition into two dynamical energies. None of these hypotheses is required for the main force, pressure, radius-work, or canonical-source results. Moving them here changes their role in the presentation, not their evidential status.

### 1. Thermal–mechanical products and a proposed kinetic/potential reading

The Schwarzschild Smarr identity can be written

$$\boxed{Mc^2 = TS + F_L L_\kappa, \qquad TS = F_L L_\kappa = \frac{Mc^2}{2}.} \tag{G1}$$

This equality holds pointwise for every stationary Schwarzschild state. No time average is required. It is a gravitational scaling balance, expressed as complementary thermal and mechanical Euler products, and motivates a stationary virial

interpretation. Static field-theory virial identities can follow from scaling an action without a time average; in general relativity the boundary terms are essential [62]. A direct test here is to scale the complete Einstein action in the same ensemble and identify its boundary balance with Eq. (100). That identification, and a separate kinetic/potential interpretation, are not proved by equality of the products alone.

For comparison, consider an ordinary bounded or periodic Hamiltonian system whose potential is homogeneous of degree $n$, $V(\lambda q) = \lambda^n V(q)$. The virial theorem gives

$$2\langle K\rangle = n\langle V\rangle, \qquad \langle K\rangle = \frac{n}{n+2}E, \qquad \langle V\rangle = \frac{2}{n+2}E, \tag{G2}$$

when the displayed averages and denominator are well defined [32]. The theorem constrains the two averaged components across a family of total energies; it does not require one to decrease whenever the other increases. For $n > 0$, raising $E$ raises both averages. A harmonic oscillator ($n = 2$) has equal halves. A complete ideal vertical projectile flight in uniform gravity, from launch at $z = 0$ to the instant it returns to $z = 0$ before any dissipative impact, has $n = 1$ and $\langle K\rangle = E/3$, $\langle V\rangle = 2E/3$: because the virial function $G = pz$ vanishes at both endpoints, the finite-time boundary term is exactly zero. Repeating the flight with elastic bounces makes the example periodic. The Kepler case $n = -1$ instead has negative binding energy, so adding positive energy makes the orbit less bound and changes the signs and magnitudes differently.

In ordinary mechanics, kinetic and potential energy are called distinct not because they are different substances, but because they are functions of different state-space data and generate different responses: kinetic energy depends on momentum or velocity and controls inertial motion, whereas potential energy depends on configuration and its gradient supplies force. They can be converted into one another while remaining operationally distinguishable energy forms.

The horizon split has an analogous, though not identical, structural basis. On the extended $(A, L_\kappa)$ source space,

$$TS = A\left(\frac{\partial U_\partial}{\partial A}\right)_{L_\kappa}, \qquad F_L L_\kappa = -L_\kappa\left(\frac{\partial U_\partial}{\partial L_\kappa}\right)_A. \tag{G3}$$

The first expression uses the area/entropy derivative and the second uses the normal-scale derivative. Both products are identically the same function $U_\partial(A, L_\kappa)$ even on the extended source space. The distinct differential legs are $\Pi_\partial\, dA$ and $-F_L\, dL_\kappa$. They do not establish independent kinetic and potential energies. Equation (28) also shows why these descriptions do not add a second scalar canonical degree of freedom.

There is an additional exact thermodynamic fact. The formal on-shell Schwarzschild Helmholtz free energy is

$$\boxed{\mathcal{F}_{\mathrm{H}}^{\mathrm{on}} = Mc^2 - TS = \frac{Mc^2}{2} = F_L L_\kappa.} \tag{G4}$$

The scale product equals the formal on-shell free-energy value in this stationary setting. This equality does not identify a separately exchangeable mechanical energy reservoir. Asymptotically flat Schwarzschild has negative heat capacity and is not a stable unconstrained canonical ensemble, but the on-shell Euclidean free-energy value remains well defined [33].

The thermal–mechanical interpretation therefore has a precise basis: $TS$ is the thermal Euler product and $F_L L_\kappa$ the scale-work product, which also equals the formal free energy in this sector. Their balance is worth testing as a stationary virial relation. A literal kinetic/potential split would additionally require separately defined observables and a derivation of their dynamical or action-scaling roles.

Along the Schwarzschild curve,

$$d(TS) = d(F_L L_\kappa) = dU_\partial = F_L\, dL_\kappa = \frac{1}{2}d(Mc^2), \qquad T\, dS = d(Mc^2). \tag{G5}$$

Adding black-hole mass increases the common value of the two Euler products by half the added mass energy. The boundary equation coexists exactly with the standard first law: the area leg contributes $2dU_\partial$ and the scale leg subtracts $dU_\partial$, leaving the complete differential $dU_\partial$.

### 2. Newtonian field-stress correspondence

**Status: a sector-specific comparison, not a derivation.** The same pressure has a familiar stationary field-stress correspondence. For a Newtonian gravitational field $\boldsymbol{g} = -\boldsymbol{\nabla}\Phi$, one conventional field-stress tensor is

$$t_{ij}^{(g)} = -\frac{1}{4\pi G}\left(g_i g_j - \frac{1}{2}\delta_{ij}|\boldsymbol{g}|^2\right). \tag{G6}$$

For a surface normal parallel to $\boldsymbol{g}$, the longitudinal compressive stress magnitude is

$$P_g^{\parallel} \equiv -t_{nn}^{(g)} = \frac{|\boldsymbol{g}|^2}{8\pi G}. \tag{G7}$$

At $R_H$, the Newtonian field magnitude is $|\boldsymbol{g}_N| = GM/R_H^2 = \kappa_\infty$, hence

$$P_g^{\parallel}(R_H) = P_L^{\mathrm{Schw}}. \tag{G8}$$

This is a match of stationary, asymptotically normalized magnitudes. It does not turn $P_L$ into a covariant local bulk stress. Rindler already shows that the boundary response is more general than a source-field interpretation [29–31].

### 3. Conditional quantum-phase interpretation

**Status: exact scale identities; conditional physical interpretation.** The main text defines $\bar{\lambda}_\partial = \hbar c/U_\partial$ and establishes its entropy relation without a microscopic model. The additional reading below requires a quantum generator realizing that boundary energy. No such independent microscopic derivation is supplied here.

The inverse-energy definition is exact; the phase interpretation is conditional. In any quantum realization in which $U_\partial$ is an energy eigenvalue of the Hamiltonian that generates the selected boundary time, Schrödinger evolution gives

$$\omega_\partial \equiv \frac{U_\partial}{\hbar}, \qquad \bar{\lambda}_\partial = \frac{c}{\omega_\partial}. \tag{G9}$$

Then $\bar{\lambda}_\partial$ is the light-travel distance associated with one radian of the corresponding quantum phase. The present macroscopic analysis does not construct a boundary Hilbert space or prove that $U_\partial$ is a sharp Hamiltonian eigenvalue. Accordingly, $\bar{\lambda}_\partial$ is rigorously a boundary inverse-energy length; the Compton-equivalent phase language records the standard energy–frequency map and does not assert a propagating boundary particle or a directly observable global oscillation.

Let

$$C_E \equiv c\Delta\tau_E = 2\pi L_\kappa, \qquad \lambda_\partial \equiv 2\pi\bar{\lambda}_\partial \tag{G10}$$

denote the Euclidean thermal circumference in light-travel units and the corresponding full boundary Compton-equivalent wavelength. Then

$$\boxed{\frac{C_E}{\bar{\lambda}_\partial} = N_S, \qquad \frac{C_E}{\lambda_\partial} = \frac{N_S}{2\pi}, \qquad \frac{\omega_\partial}{\omega_1^{\mathrm{E}}} = \frac{N_S}{2\pi}.} \tag{G11}$$

Because $\bar{\lambda}_\partial$ is a reduced, one-radian inverse-energy length, the first ratio counts such reduced intervals and the second counts full $2\pi$ intervals. If a quantum boundary Hamiltonian realizes $U_\partial$, these are respectively phase radians and full phase cycles. Without that additional microscopic realization, the exact statement is the ratio of the Euclidean thermal period to the inverse-energy time $\hbar/U_\partial$. Analytically continued around one thermal period, the same statement is

$$\boxed{\frac{U_\partial \Delta\tau_E}{\hbar} = \beta U_\partial = \frac{S}{k_B} = N_S.} \tag{G12}$$

This is the dimensionless Euclidean Boltzmann exponent associated with the boundary potential over one thermal period. Equivalently, $S/k_B$ is the boundary energy measured in thermal units $k_B T$. If $U_\partial$ is realized as a quantum time-translation generator, Wick rotation turns its Lorentzian phase factor into this Euclidean decay exponent, so the same quantity may also be read as the boundary-energy phase advance continued around the thermal circle. It is not a claim that $\beta U_\partial$ equals the complete on-shell gravitational action or that the horizon is literally tiled by independent Compton cells. The model-independent content is that entropy measures the separation between the boundary inverse-energy scale and the Euclidean thermal scale.

## Appendix H: Conditional Planck-scale benchmarks

**Status: formal extrapolations and additional scale-matching conditions.** The following configurations are retained as interesting connections, not as part of the derivation of $F_L$ or $P_L$. Their numerical values follow after

imposing the displayed equalities on the Einstein boundary and thermal formulas. No separate quantum-gravity derivation selects these equalities or shows that the classical and semiclassical formulas remain exact at the Planck scale. The $G/2$ comparison below is a normalization counterfactual, not a proposed modification of gravity.

The exact ratio in Eq. (86) permits one to *choose* the scale-crossing condition $\bar{\lambda}_\partial = L_\kappa$. With that additional condition, the formal crossing has the following properties. Any frequency language uses the conditional interpretation in Appendix G 3.

At

$$A = 8\pi\ell_{\rm P}^2, \qquad S = 2\pi k_B, \tag{H1}$$

the two reduced lengths and the two frequencies coincide:

$$\boxed{\bar{\lambda}_\partial = L_\kappa, \qquad \omega_\partial = \omega_1^{\rm E}, \qquad U_\partial = \hbar\omega_1^{\rm E} = 2\pi k_B T.} \tag{H2}$$

Equivalently, one full boundary Compton-equivalent wavelength $2\pi\bar{\lambda}_\partial$ fits around the Euclidean thermal circumference $2\pi L_\kappa$. This is a formal quantum–horizon scale-crossing benchmark: the inverse-energy scale of the complete boundary potential is no longer parametrically separated from its geometric and thermal scale. It is not evidence by itself for a phase transition, nor does it locate a physical transition. It identifies the absence of a large separation between the two lengths within the continued formulas.

For Schwarzschild, $M_\partial = M/2$, so $\bar{\lambda}_\partial = 2\hbar/(Mc)$, while $L_\kappa = 2R_H$. Consequently $\bar{\lambda}_\partial = L_\kappa$ is exactly equivalent to the conventional reduced-Compton condition $\hbar/(Mc) = R_H$. With the standard definitions this occurs at

$$M = \frac{m_{\rm P}}{\sqrt{2}}, \qquad R_H = \sqrt{2}\,\ell_{\rm P}, \qquad L_\kappa = \bar{\lambda}_\partial = 2\sqrt{2}\,\ell_{\rm P}. \tag{H3}$$

The familiar statement that the Compton and Schwarzschild scales meet "at the Planck scale" therefore carries order-one convention factors: using the gravitational radius $GM/c^2$ instead of $R_H = 2GM/c^2$, or the ordinary rather than reduced Compton wavelength, moves the numerical crossing. The present boundary variables reproduce the conventional reduced-Compton–Schwarzschild-radius crossing because both lengths are doubled.

*a. Planck units and the status of ultraviolet benchmarks.* The Planck quantities used here are natural combinations of the constants that normalize relativity, quantum mechanics, and classical gravity:

$$\ell_{\rm P} = \sqrt{\frac{\hbar G}{c^3}}, \qquad t_{\rm P} = \frac{\ell_{\rm P}}{c}, \qquad m_{\rm P} = \sqrt{\frac{\hbar c}{G}}, \qquad E_{\rm P} = m_{\rm P}c^2, \tag{H4}$$

$$a_{\rm P} \equiv \frac{c}{t_{\rm P}} = \frac{c^2}{\ell_{\rm P}}, \qquad F_{\rm P} \equiv \frac{E_{\rm P}}{\ell_{\rm P}} = \frac{c^4}{G}. \tag{H5}$$

They are not arbitrary symbols: for example, the dimensionless quantum-gravitational coupling constructed from an energy scale $E$ is

$$\alpha_G(E) \equiv \frac{GE^2}{\hbar c^5} = \left(\frac{E}{E_{\rm P}}\right)^2, \tag{H6}$$

so the Planck energy is where this simple coupling estimate becomes order unity. The definitions do not establish that $\ell_{\rm P}$ is an exact minimum length, that $a_{\rm P}$ is a maximum acceleration, or that the unmodified Einstein and Hawking coefficients remain exact at those scales. General relativity is a controlled quantum effective field theory well below the Planck scale, while its ultraviolet completion is unknown [5]. The relations below are consequently benchmarks obtained by propagating the present Einstein boundary and Hawking–Unruh normalizations, not predictions of completed quantum gravity.

The Rindler relation $L_\kappa = c^2/a$ is exact within classical Minkowski geometry. Inserting $a = a_{\rm P}$ gives $L_\kappa = \ell_{\rm P}$ by definition. This does not demonstrate a realizable Planck-accelerated detector or a universal acceleration bound; it identifies the point at which the classical Rindler proper distance reaches the Planck length.

*b. A hierarchy of distinct Planck benchmarks.* The least restrictive and most clock-invariant criterion internal to the present framework is the loss of scale separation,

$$\boxed{\bar{\lambda}_\partial = L_\kappa.} \tag{H7}$$

It fixes the transverse data

$$A = 8\pi\ell_{\rm P}^2, \qquad R = \sqrt{2}\,\ell_{\rm P} \quad \text{for a spherical cut,} \tag{H8}$$

but it does not fix the common value of $\bar{\lambda}_\partial$ and $L_\kappa$. The reason is that a constant rescaling of the selected normal clock rescales $U_\partial$ and inversely rescales $\bar{\lambda}_\partial$ and $L_\kappa$, while leaving $A$ unchanged. A physical sector supplies the remaining relation: Cai–Kim gives $L_\kappa = R = \sqrt{2}\,\ell_{\rm P}$, whereas Schwarzschild gives $L_\kappa = 2R_H = 2\sqrt{2}\,\ell_{\rm P}$ at the same crossover area. The area is common; the normal geometry is sector dependent.

A stronger, maximally symmetric source-space benchmark adds the Planck normal scale,

$$\boxed{L_\kappa = \bar{\lambda}_\partial = \ell_{\rm P}.} \tag{H9}$$

Equations (H1) and (H9) then give

$$\boxed{A = 8\pi\ell_{\rm P}^2, \quad R = \sqrt{2}\,\ell_{\rm P}, \quad U_\partial = E_{\rm P}, \quad M_\partial = m_{\rm P}, \quad F_L = F_{\rm P}, \quad k_B T = \frac{E_{\rm P}}{2\pi}.} \tag{H10}$$

This is the simplest all-Planck alignment in the extended $(A, L_\kappa)$ boundary-source space. It is not the conventional Schwarzschild crossover and it is not the Cai–Kim relation $L_\kappa = R$: for a spherical cut it has $R/L_\kappa = \sqrt{2}$.

A different, equally simple but sector-specific benchmark is to impose a Planck areal radius in the Cai–Kim relation,

$$R = L_\kappa = \ell_{\rm P}. \tag{H11}$$

Keeping the standard Einstein normalization then yields

$$\boxed{A = 4\pi\ell_{\rm P}^2, \quad U_\partial = \frac{E_{\rm P}}{2}, \quad M_\partial = \frac{m_{\rm P}}{2}, \quad F_L = \frac{F_{\rm P}}{2}, \quad \bar{\lambda}_\partial = 2\ell_{\rm P}, \quad S = \pi k_B.} \tag{H12}$$

There is no inconsistency: the condition $R = \ell_{\rm P}$ fixes only half of the crossover area. Requiring in addition $U_\partial = E_{\rm P}$ and $F_L = F_{\rm P}$ at $R = L_\kappa = \ell_{\rm P}$ would double the Einstein boundary coefficient, equivalently replacing $G$ by an effective $G/2$ in this one-scale benchmark. No established Planck-regime principle presently selects that modification.

These choices answer different conditional questions: $\bar{\lambda}_\partial = L_\kappa$ compares the two scales defined by the framework; $L_\kappa = \bar{\lambda}_\partial = \ell_{\rm P}$ adds a symmetric all-Planck benchmark; and $R = L_\kappa = \ell_{\rm P}$ specifies a separate Cai–Kim-like Planck-radius continuation. None is claimed to be the unique state chosen by quantum gravity.

## Appendix I: Reproducible local Einstein–Hilbert derivation in Gaussian-null coordinates

This appendix derives the selected scalar source response directly from the Einstein–Hilbert (EH) boundary current. It makes explicit the local metric calculation, boundary functionals, endpoint contribution, and clock conversion underlying the canonical discussion in the main text. The result is a restricted local calculation with a specified action prescription and explicit geometric and variational premises.

### 1. Domain, orientation, and metric variations

Initially set $c = 1$ and write $C = (8\pi G)^{-1}$. On a smooth null patch $\mathcal{N} = \{r = 0\}$, use Gaussian-null coordinates $(u, r, x^A)$, $A = 1, 2$, with

$$ds^2 = -2r\widehat{\kappa}\, du^2 + 2\, du\, dr + 2r\beta_A\, du\, dx^A + q_{AB}\, dx^A dx^B. \tag{I1}$$

Only the metric and its first derivatives at $r = 0$ enter the calculation. The cut metric $q_{AB}$ is a general positive-definite symmetric matrix; its $u$, $r$, and transverse first derivatives are retained. The generator is $\ell = \partial_u$, with $\nabla_\ell \ell = \widehat{\kappa}\ell$ on $\mathcal{N}$. The null location, generator, coordinate identifications, normalization $g_{ur} = 1$, and integration domain $[u_i, u_f] \times S$ are fixed. We orient the boundary current with the normal covector $-dr$. Reversing orientation reverses the entire boundary response consistently.

Let $h_{ab} = \delta g_{ab}$ preserve this chart and null location. On $\mathcal{N}$, its only nonzero metric components are $h_{AB} = \delta q_{AB}$; the independent radial derivatives include $\partial_r h_{uu} = -2\delta\widehat{\kappa}$ and $\partial_r h_{uA} = \delta\beta_A$. All first derivatives of $h_{AB}$ are retained. A dot denotes $\partial_u$, and $Q = q^{-1}$. Define

$$\begin{aligned} \epsilon_S &= \sqrt{\det q}\, d^2x, & \delta\epsilon_S &= \tfrac{1}{2}\epsilon_S \operatorname{tr}(Qh), \\ \theta &= \tfrac{1}{2}\operatorname{tr}(Q\dot{q}), & \delta\theta &= \tfrac{1}{2}\operatorname{tr}(Q\dot{h} - QhQ\dot{q}), \\ \sigma_{AB} &= \tfrac{1}{2}\dot{q}_{AB} - \tfrac{1}{2}\theta q_{AB}, & \mu &= \widehat{\kappa} + \tfrac{1}{2}\theta. \end{aligned} \tag{I2}$$

The variation commutes with $\partial_u$ in this fixed domain. No field equation, spherical symmetry, or stationary cut metric is used to obtain the local current below.

### 2. Direct current and the selected boundary action

For the EH action, the un-oriented current component is

$$v^r = g^{ab}\delta\Gamma^r{}_{ab} - g^{rb}\delta\Gamma^a{}_{ab} = -2\delta\widehat{\kappa} - \operatorname{tr}(Q\dot{h}) + \tfrac{1}{2}\operatorname{tr}(QhQ\dot{q}). \tag{I3}$$

For example, this follows by substituting the metric and variation jets into

$$\delta\Gamma^a{}_{bc} = \frac{1}{2}\delta g^{ad}(\partial_b g_{dc} + \partial_c g_{db} - \partial_d g_{bc}) + \frac{1}{2}g^{ad}(\partial_b h_{dc} + \partial_c h_{db} - \partial_d h_{bc}), \qquad \delta g^{ab} = -g^{ac}h_{cd}g^{db}. \tag{I4}$$

Equivalently, the current can be calculated from $v^r = \nabla_b h^{rb} - \nabla^r(g^{ab}h_{ab})$, with $h^{ab} = g^{ac}g^{bd}h_{cd}$. Both expressions retain general cut-metric first jets; the cancellation of the $\beta_A$ and transverse/radial cut jets in Eq. (I3) is a result of this fixed-generator projection, not a proof that generator responses vanish for more general variations.

The oriented contribution is $\Theta_{\mathrm{EH}}|_{\mathcal{N}} = -(C/2)\int \epsilon_S v^r\, du$. Using Eq. (I2) gives

$$\Theta_{\mathrm{EH}}|_{\mathcal{N}} = C\int_{\mathcal{N}} \Big[-\mu\,\delta\epsilon_S + \frac{\epsilon_S}{2}\sigma^{AB}h_{AB} + \delta\{\epsilon_S(\widehat{\kappa} + \theta)\}\Big]\, du. \tag{I5}$$

This agrees with the fixed-generator scalar/shear part of the null decomposition of Ref. [11], Eqs. (4.27) and (5.3)–(5.5) in the arXiv v2 numbering. That reference separately retains its corner terms. Equation (I5) was derived here from the local EH current; those global terms have not been inferred to vanish.

Specify the local boundary functionals

$$\begin{aligned} B_0 &= C\int_{\mathcal{N}} \epsilon_S(\widehat{\kappa} + \theta)\, du, & B_\mu &= C\int_{\mathcal{N}} \epsilon_S\mu\, du, \\ I_{\mathrm{rate}} &= I_{\mathrm{EH}} - B_0 + B_\mu. \end{aligned} \tag{I6}$$

The last line selects this part of the action prescription; any remaining boundaries and matter/corner terms require their own controlled data. Direct variation, rather than a choice of energy function alone, now yields

$$\Theta_{\rm rate}|_{\mathcal{N}} = C\int_{\mathcal{N}} \epsilon_S\,\delta\mu\,du + \Theta_{\rm shear}, \qquad \Theta_{\rm shear} = \frac{C}{2}\int_{\mathcal{N}} \epsilon_S \sigma^{AB} h_{AB}\,du. \tag{I7}$$

The scalar exchange is $-C\mu\delta\epsilon_S + \delta(C\epsilon_S\mu) = C\epsilon_S\delta\mu$. Its field-space exterior derivative is the same in either polarization, $C\delta\epsilon_S \wedge \delta\mu$. Conservative fixed-source problems may set $\delta\mu = 0$; the displayed source response compares nearby such problems with other response channels controlled.

### 3. Endpoint area and tangent nonexpansion

The action shift in Eq. (I6) has a simple exact endpoint form. Since $\partial_u\epsilon_S = \theta\epsilon_S$ and the cut domain is fixed,

$$\begin{aligned} I_{\rm rate} - I_{\rm EH} = B_\mu - B_0 &= -\frac{C}{2}\int_{\mathcal{N}} \epsilon_S\theta\,du = -\frac{C}{2}(A_f - A_i), \\ \delta(I_{\rm rate} - I_{\rm EH}) &= -\frac{C}{2}(\delta A_f - \delta A_i), \qquad A(u) = \int_S \epsilon_S(u). \end{aligned} \tag{I8}$$

Fixed endpoint coordinates do not by themselves set this variation to zero. Impose instead the background and tangent restrictions

$$\theta = 0, \qquad \delta\theta = 0 \quad \text{throughout } [u_i, u_f]. \tag{I9}$$

Then $\partial_u\delta\epsilon_S = \delta(\theta\epsilon_S) = \epsilon_S\delta\theta + \theta\delta\epsilon_S = 0$, so $\delta A_f = \delta A_i$. Each endpoint area may have a nonzero variation. Thus the variation of this particular endpoint difference vanishes on the tangent sector, and $\delta\mu = \delta\widehat{\kappa}$ there. With the shear response separately fixed or accounted for, the scalar result is

$$\Theta^{(0)}_\kappa = C\int_{\mathcal{N}} \epsilon_S\,\delta\widehat{\kappa}\,du. \tag{I10}$$

This argument controls the endpoint term actually present in Eq. (I8). It does not eliminate unrelated corner responses. A useful stronger subcase is $h_{AB} = \dot{h}_{AB} = 0$: Eq. (I3) then gives Eq. (I10) directly from EH, even before the boundary-functional exchange.

### 4. Physical clock, length work, and the complementary corner check

Restore the EH action normalization $c^3/(16\pi G)$ and choose the fixed physical-clock dictionary $du = c\,d\tau$, $\widehat{\kappa} = \kappa/c^2$, with $c$ and $G$ fixed. Both factors are needed:

$$\frac{c^3}{8\pi G}\int \epsilon_S\,\delta(\kappa/c^2)\,c\,d\tau = \frac{c^2}{8\pi G}\int d\tau\int_S \epsilon_S\,\delta\kappa. \tag{I11}$$

The endpoint shift in physical action units is $-c^3(A_f - A_i)/(16\pi G)$. For uniform stationary data on a fixed duration $\Delta\tau$, take the positive branch $\kappa > 0$ and $L_\kappa = c^2/\kappa > 0$. It gives

$$\frac{\Theta^{(0)}_\kappa}{\Delta\tau} = \frac{Ac^2}{8\pi G}\delta\kappa = -\frac{Ac^4}{8\pi G L_\kappa^2}\delta L_\kappa = -F_L\delta L_\kappa, \qquad P_L = \frac{F_L}{A} = \frac{\kappa^2}{8\pi G}. \tag{I12}$$

This is the selected source one-form. The complete independent-coordinate energy differential remains

$$\delta U_\partial = \Pi_\partial\delta A - F_L\delta L_\kappa, \qquad U_\partial = \frac{Ac^4}{8\pi G L_\kappa}, \qquad \Pi_\partial = \frac{c^4}{8\pi G L_\kappa}. \tag{I13}$$

If the specified on-shell action and admissible stationary source family exist with the remaining responses controlled, the reduced Hamilton–Jacobi statement is $\partial I_{\rm os}/\partial L_\kappa = -\Delta\tau F_L$ at fixed other sources. The local computation does not prove existence of arbitrary such families. Fixed clock prescription also does not mean $\delta\kappa = 0$.

The canonical role of $F_L$ is precisely its coefficient in this allowed length-source variation. It does not require $U_\partial$ to equal the entire Hamiltonian of the spacetime or boundary-value problem. That Hamiltonian depends on the selected

evolution and its full boundary and reference prescription. The potential, selected response one-form, and complete Hamiltonian variation therefore retain distinct definitions.

For the consistently oriented stationary area–boost pair already discussed in the main text, $J_\eta = Ac^3/(8\pi G)$ and $\eta = c\Delta\tau/L_\kappa$ give $J_\eta \delta\eta/\Delta\tau = -F_L \delta L_\kappa$. This is a complementary coefficient check with the same area normalization, clock, and fixed duration. Similarly, $\delta A \wedge \delta\Pi_\partial = \delta L_\kappa \wedge \delta F_L$ checks the coordinate change of the scalar symplectic two-form; it does not introduce an additional independent canonical pair.

## 5. Restrictions and counterexamples

The following examples establish why the restrictions used above are needed:

1. *Expansion tangency.* Even on a background with $\theta = 0$, the rate form retains $C\epsilon_S \delta\theta/2$ unless $\delta\theta = 0$. A fixed coordinate interval with unequal endpoint area variations likewise retains the endpoint term in Eq. (I8).

2. *Shear.* At a cut choose $q = \mathbb{1}$, $\dot{q} = \mathrm{diag}(2s, -2s)$, $h = \mathrm{diag}(h_0, -h_0)$, and $\dot{h} = \mathrm{diag}(2sh_0, 2sh_0)$. These jets give $\theta = \delta\theta = 0$, but leave $C\epsilon_S s h_0$ in the EH response beyond $C\epsilon_S \delta\widehat{\kappa}$. This is a local kinematic counterexample; it makes no claim about existence of an on-shell nonexpanding solution with additional energy conditions. Such extra conditions may restrict shear, but are not silently part of this current calculation.

3. *Full energy variation.* The difference between $\delta U_\partial$ and the selected length one-form is $\Pi_\partial \delta A$, which need not vanish.

4. *Boundary prescription.* Adding a nonzero local functional $\alpha A\widehat{\kappa}$ shifts the coefficient of $\delta\widehat{\kappa}$ by $\alpha A$, even if the same algebraic function $U_\partial$ is retained. Its exact variation leaves the two-form unchanged. Thus $U_\partial$ alone does not uniquely determine the action response.

The demonstrated result is therefore the EH scalar response in the fixed Gaussian-null domain with the specified boundary functionals, orientation, physical clock, and tangent nonexpansion. Remaining shear, generator, matter, other-boundary, and corner responses must be controlled in the chosen variational problem. No arbitrary-generator covariance theorem, moving-cut or global-corner theorem, or canonical certification of a generic evolving FRW worldtube follows from these local checks. These limits leave the derived coefficient and its length-work interpretation intact in the declared sector.


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