

# Horizon Boundary Mechanics: Force, Pressure, and Work from the Einstein Boundary Potential

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## Abstract

Recognizing acceleration length as a geometrically normalized work coordinate of the established Einstein boundary potential identifies a new generalized boundary force and normal-scale pressure. These join the area-conjugate surface stress in one macroscopic state law. The selected nonexpanding-null action response supplies the principal canonical justification; the stationary area-boost route gives a complementary interpretation with shared clock and area normalization. Schwarzschild work becomes  $M\kappa dR = P_R dV$ , with  $P_R = 2P_L$ . The separately assigned instantaneous equilibrium FRW screen has  $P_R = P_L = \rho/3$  and potential equal to the Misner–Sharp horizon energy. Temperature and entropy enter after the mechanics. The result concerns a specified boundary prescription, not a new interaction or an unrestricted moving-horizon theorem.

**Keywords:** horizon boundary mechanics, gravitational boundary response, acceleration length, null symplectic structure, FRW cosmology

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## 1. Geometry and the boundary potential

A horizon cut has transverse area  $A$  and a surface-gravity scale  $\kappa$ , with its physical time normalization specified. Express the same acceleration scale as a length:

$$L_s = \frac{c^2}{\kappa}, \quad L_\kappa = |L_s|, \quad \kappa \neq 0. \quad (1)$$

In Rindler geometry  $L_\kappa$  is the normal observer-horizon proper distance on the instantaneous rest slice. More generally, the characteristic time  $t_\kappa = c/|\kappa|$  gives the light-travel length  $L_\kappa = ct_\kappa$ . For stationary normal flow,  $t_\kappa$  is the time for one unit of boost rapidity. It is not an assumed photon transit time from the horizon;  $L_\kappa$  is not a universal spherical radius.

For the Einstein boundary prescription used here,

the energy-valued potential is [5, 8, 9]

$$U_\partial(A, L_s) = \frac{A\kappa c^2}{8\pi G} = \frac{c^4}{8\pi G} \frac{A}{L_s}. \quad (2)$$

The Einstein boundary structure fixes this normalization (Sec. 3). The coefficient  $c^4/(8\pi G) = 1/\kappa_E$ , with  $\kappa_E = 8\pi G/c^4$ , has units of energy per length;  $A/L_s$  supplies the length factor. Dimensional analysis alone does not fix the potential.

The mechanical question is what displacement pairs with the potential's work. Recasting its surface-gravity dependence through  $L_s$  identifies a force conjugate to horizon-scale change alongside the familiar area response. Here  $U_\partial = Mc^2/2$  for Schwarzschild and  $U_\partial = E_{\text{MS}}$  for the equilibrium-FRW screen.

## 2. One differential, common mechanical responses

At fixed  $c$  and  $G$ , differentiating before imposing a particular horizon family's relation between  $A$  and

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$L_s$  gives

$$\begin{aligned} dU_\partial &= \frac{c^4}{8\pi GL_s} dA - \frac{Ac^4}{8\pi GL_s^2} dL_s \\ &= \Pi_\partial dA - F_L dL_s. \end{aligned} \quad (3)$$

Writing  $M_\partial = U_\partial/c^2$  for the potential's mass equivalent, the coefficients are

$$\Pi_\partial = \left( \frac{\partial U_\partial}{\partial A} \right)_{L_s} = \frac{U_\partial}{A} = \frac{\kappa c^2}{8\pi G}, \quad (4)$$

$$F_L = - \left( \frac{\partial U_\partial}{\partial L_s} \right)_A = \frac{U_\partial}{L_s} = M_\partial \kappa. \quad (5)$$

The established area stress and the new length-conjugate force follow from the same specified potential. The corresponding new normal-scale pressure is the force per area,

$$P_L = \frac{F_L}{A} = \frac{\Pi_\partial}{L_s} = \frac{c^4}{8\pi GL_s^2} = \frac{\kappa^2}{8\pi G}. \quad (6)$$

These are two conjugate coefficients and one response per area, not three independent canonical pairs. The mass-acceleration form is an equivalent identity, not an additional material equation of motion.

The mechanical energy identity and differential are

$$\begin{aligned} U_\partial &= \Pi_\partial A = F_L L_s = P_L A L_s, \\ dU_\partial &= \Pi_\partial dA - F_L dL_s. \end{aligned} \quad (7)$$

Since  $U_\partial = A\Pi_\partial$ , the same second contribution is  $A d\Pi_\partial = -F_L dL_s$ , so  $d\Pi_\partial = -P_L dL_s$ . Pressure is therefore also the negative stress slope. This is a comparison of boundary configurations, not a material-layer pressure gradient. The geometric scale laws

$$\frac{1}{L_s} = \kappa_E \Pi_\partial, \quad \frac{1}{L_s^2} = \kappa_E P_L \quad (8)$$

are consequences of the common mechanics, not its starting hierarchy.

No enclosed volume is needed. If  $\vartheta_V = A dL_s$ ,

$$F_L dL_s = P_L \vartheta_V, \quad d\vartheta_V = dA \wedge dL_s. \quad (9)$$

Thus  $\vartheta_V$  is not generally a global volume differential on the two-coordinate source space. The unchanged  $P_L$  multiplies  $dV$  only where  $dV = A dL_s$ . A spherical family instead supplies  $dV = A dR$ ; transforming the conjugate pressure gives exact radial volume work, as shown below. The product  $AL_s$  is not generally that volume.

### 3. Canonical foundation of the response

The gravitational action variation gives interior field equations and a boundary one-form  $\Theta$ . This symplectic potential is linear in a field variation, like  $p \delta q$ ; it is not the energy function  $U_\partial$ . Its field-space exterior derivative is the canonical two-form. A choice of boundary terms determines the displayed source and response.

Let a null hypersurface  $\mathcal{N}$  be foliated by spacelike two-dimensional cuts  $S$ , with metric  $q_{AB}$  and area element  $\epsilon_S = \sqrt{q} d^2x$ . For a selected null generator  $\xi^a$ , define

$$\nabla_\xi \xi^a = \widehat{\kappa} \xi^a, \quad \theta = \frac{1}{\sqrt{q}} \mathcal{L}_\xi \sqrt{q}. \quad (10)$$

Inaffinity  $\widehat{\kappa}$  measures nonaffine generator parameterization; expansion  $\theta$  measures fractional area change. With a length parameter both have inverse-length units, and  $\kappa = c^2 \widehat{\kappa}$ . A null generator has no proper time, so its inaffinity is not a material observer's proper acceleration.

Hopf-müller and Freidel identify the scalar canonical combination [8, 9]

$$\mu = \widehat{\kappa} + \frac{1}{2} \theta. \quad (11)$$

Shear and generator/twist data occupy other sectors. In units  $c = 1$ , the area polarization has scalar term  $\Theta_A^{(0)} = -(8\pi G)^{-1} \int \mu \delta \epsilon_S dv$ . Adding the exact variation of  $B_\mu = (8\pi G)^{-1} \int \mu \epsilon_S dv$  gives

$$\Theta_\mu^{(0)} = \Theta_A^{(0)} + \delta B_\mu = \frac{1}{8\pi G} \int_{\mathcal{N}} \epsilon_S \delta \mu dv. \quad (12)$$

This is  $-p \delta q + \delta(pq) = q \delta p$ : a change of polarization, leaving the two-form unchanged. Covariant mixed boundary conditions identify the same

derivative-containing scalar source [10]. Orientation, physical generator, cut identifications, and other boundary/corner data are specified.

Restrict to the nonexpanding sector and variations tangent to it:

$$\theta = 0, \quad \delta\theta = 0. \quad (13)$$

Restoring acceleration units and the prescribed physical time label  $\tau$  gives

$$\Theta_\kappa^{(0)} = \frac{c^2}{8\pi G} \int d\tau \int_S \epsilon_S \delta\kappa. \quad (14)$$

The coordinate change follows directly from  $L_s = c^2/\kappa$ :  $\delta\kappa = -c^2\delta L_s/L_s^2$ . Substituting it into the gravitational response gives

$$\begin{aligned} \Theta_L^{(0)} &= - \int d\tau \int_S \epsilon_S \frac{c^4}{8\pi G L_s^2} \delta L_s \\ &= - \int d\tau \int_S \epsilon_S P_L \delta L_s. \end{aligned} \quad (15)$$

For a common length variation,  $F_L = \int_S P_L dA$ ; for a uniform finite cut,  $F_L = AP_L$  and  $\Theta_L^{(0)} = - \int d\tau F_L \delta L_s$ . This is the coefficient already obtained from the potential, now identified in the selected Einstein action. Pressure is the local coefficient; force is its area integral. The area measure remains in the one-form, so  $(L_s, P_L)$  is not an additional independent pair.

Equivalently, the uniform stationary response is  $\Theta_\kappa^{(0)}/\Delta\tau = A\delta\Pi_\partial$ . The source exchange  $-\Pi_\partial\delta A + \delta(A\Pi_\partial) = A\delta\Pi_\partial$  is the product rule for the same boundary structure. It is not a second independent proof and neither selected one-form is the full differential of  $U_\partial$ .

An admissible source variation compares nearby boundary-value problems while controlling the remaining data. Within one conservative problem the prescribed mixed source is fixed. The variation must preserve the stated nullness and nonexpanding conditions and account for matter and other symplectic responses. Fixing the clock prescription does not impose  $\delta\kappa = 0$ : the geometry can change with that prescription held constant.

Equivalently, the mixed on-shell action has local Hamilton–Jacobi response

$$F_L(\tau) = -\frac{\delta I_{os}}{\delta L_s(\tau)}. \quad (16)$$

The time measure makes this derivative a force. A constant-source derivative across a fixed interval instead equals  $-\Delta\tau F_L$ . This response depends on the specified action and counterterm prescription; arbitrary source-dependent additions need not preserve it. No existence theorem for every freely assigned boundary source is claimed.

In this scalar sector,

$$\frac{\Omega^{(0)}}{\Delta\tau} = \delta A \wedge \delta\Pi_\partial = \delta L_s \wedge \delta F_L. \quad (17)$$

These are representations of one pair. The potential defines the new generalized force; the selected action establishes its source-response role. The Legendre term selects rate polarization;  $\kappa \mapsto c^2/\kappa$  is a coordinate change within it. Among lengths constructed only from  $c$  and  $\kappa$ , dimensional analysis gives  $Cc^2/\kappa$ ; Rindler distance fixes  $C = 1$ . Other coordinates preserve work when their coefficients transform. The preference is geometric, not a prohibition on reparameterization.

### 3.1. Stationary area–boost interpretation

A spacelike corner carries an area–boost pair [11, 12]. Choose its canonical orientation consistently with Eq. (15). The action-valued coefficient and the rate of the dimensionless normal boost angle are

$$J_\eta = \frac{Ac^3}{8\pi G}, \quad \omega_s = \frac{\kappa}{c} = \frac{c}{L_s}. \quad (18)$$

The boost changes the time–normal frame, not the horizon’s spatial rotation. For the same prescribed stationary clock, energy is the generator per boost times the boost rate:

$$U_\partial = J_\eta \omega_s = \frac{J_\eta}{L_s/c}, \quad U_\partial L_s = cJ_\eta. \quad (19)$$

The action coefficient contains  $c^3$ ; multiplication by  $c/L_s$  supplies the fourth power of  $c$ . This uses the

framework's normalized  $L_s$  directly. It introduces no second length or observer-energy assignment.

At fixed stationary duration  $\Delta\tau$ ,

$$\eta = \frac{c\Delta\tau}{L_s}, \quad \delta\eta = -\frac{c\Delta\tau}{L_s^2}\delta L_s. \quad (20)$$

The corner one-form therefore gives

$$\frac{J_\eta\delta\eta}{\Delta\tau} = -\frac{cJ_\eta}{L_s^2}\delta L_s = -F_L\delta L_s. \quad (21)$$

This checks the same scalar coefficient with shared Einstein area normalization, physical clock and stationary boost relation; it is not an independent empirical test. An oriented corner action and its selected canonical one-form are not interchangeable with a complete Hamiltonian variation. Nor is a boost rate prescribed on an arbitrary moving cut automatically the rate of that horizon's evolution.

### 3.2. Scope of the certification

Stationary Rindler and Schwarzschild realize the stated construction, as does a consistently normalized stationary de Sitter horizon. The cosmological constant does not remove the Einstein area–rate boundary term: at fixed  $\Lambda$ , its bulk Lagrangian contribution contains no metric derivatives and adds no derivative symplectic-potential term. A generic FRW apparent-horizon worldtube need not be null: one vanishing null expansion on each cut does not imply a nonexpanding tube. Flat dust FRW, for example, has  $R_A = 3ct/2$  and horizon speed relative to comoving flow  $c/2$ ; its tube is timelike.

On an expanding null boundary the scalar source is  $\mu$ , not just  $\widehat{\kappa}$ . Rewriting its  $\delta\kappa$  contribution retains an additional expansion contribution; it does not remove the other sectors. The FRW identities below instead use their explicit CK calibration. This is a distinction in canonical provenance, not a failure of their stated algebra.

## 4. Boundary potential versus mass first law

The usual mass first law is a Hamiltonian variation for a specified evolution, not the full differential of  $A_\kappa$ . Covariant charge variations include a symplectic-potential subtraction and generator corrections when required. Consequently

$d(TS) = T dS + S dT$  does not require adding  $S dT$  to the mass law. Under the thermal dictionary, the source exchange is exactly  $-T\delta S + \delta(TS) = S\delta T$ . Surface-Hamiltonian treatments establish the complementary  $T\delta S$  and  $S\delta T$  contributions [5, 4]. Here the rate contribution is identified as a new force–length response and pressure, with an explicit geometric work coordinate.

Where the positive-branch thermal identification applies, differentiating  $U_\partial = \omega_s J_\eta$  gives

$$\begin{aligned} \omega_s dJ_\eta &= \Pi_\partial dA = T dS, \\ J_\eta d\omega_s &= -F_L dL_\kappa = S dT. \end{aligned} \quad (22)$$

The scale leg is not generally a separate state energy: its exterior derivative is  $(c^2/8\pi G)\delta A \wedge \delta\kappa$ . The complete  $dU_\partial$  is exact. On a one-dimensional physical family the isolated work term is integrable along the path, without acquiring universal independence.

For Schwarzschild,  $E = Mc^2 = 2U_\partial$  and the constrained family gives

$$dE = T dS = 2F_L dL_\kappa, \quad dU_\partial = F_L dL_\kappa. \quad (23)$$

Appending the boundary scale term as an independent addition to  $dE = T dS$  would double-count that variation.

## 5. Stationary and cosmological realizations

The geometric dictionary is sector-specific: Rindler has independent planar area and normal length; Schwarzschild uses the infinity-normalized clock and  $U = Mc^2/2$ ; static de Sitter uses positive-magnitude  $L = R_A$ . Evolving FRW is a separately assigned screen-state realization. On a spherical marginal horizon,  $A = 4\pi R^2$  and the Misner–Sharp energy is  $E_{\text{MS}} = c^4 R/(2G)$  [15, 16]. The area potential therefore has the spherical form

$$U_\partial = E_{\text{MS}} \frac{R}{L_s}, \quad U_\partial L_s = cJ_\eta = E_{\text{MS}} R.$$

For  $\kappa > 0$ ,  $L_s = L_\kappa$ : the ratio  $U_\partial/E_{\text{MS}} = R/L_\kappa$  is one in Cai–Kim FRW and one half in standard Schwarzschild. This is a realization, not a replacement for the area potential. The horizon formula is not the energy of an arbitrary sphere, and  $E_{\text{MS}} \propto R$  must be retained in any differential.

### 5.1. Rindler: the acceleration-length interpretation

For an observer with constant proper acceleration  $a$  in Minkowski spacetime,

$$\kappa = a, \quad L_\kappa = c^2/a, \quad P_L = a^2/(8\pi G). \quad (24)$$

The length is the normal observer–horizon proper distance on the instantaneous rest slice; transverse patch area  $A = \int dy dz$  is independent. No source mass or spherical projection is required. Related energy constructions exist [13]; here rate, length, and potential use the single normalization of Eq. (1).

### 5.2. Schwarzschild: work of horizon-size change

With stationary Killing time normalized at infinity and areal radius  $R_H$  defined by  $A_H = 4\pi R_H^2$ ,

$$\kappa = \frac{c^2}{2R_H}, \quad L_\kappa = 2R_H, \quad F_L = \frac{c^4}{8G}, \quad U_\partial = \frac{Mc^2}{2}. \quad (25)$$

The Smarr relation becomes

$$Mc^2 = TS + F_L L_\kappa, \quad TS = F_L L_\kappa = \frac{Mc^2}{2}. \quad (26)$$

The products coincide pointwise and represent different derivative directions of one potential, not independently stored energies. The same work expressed through the actual horizon size is

$$\begin{aligned} F_R &= 2F_L = \frac{U_\partial}{R_H} = M\kappa, \\ P_R &= \frac{F_R}{A_H} = 2P_L = \frac{\kappa^2}{4\pi G}, \\ F_L dL_\kappa &= F_R dR_H = P_R dV_H. \end{aligned} \quad (27)$$

Here  $V_H = 4\pi R_H^3/3$  and  $dV_H = A_H dR_H$ . Since  $dL_\kappa/L_\kappa = dR_H/R_H$ , length work measures the same fractional horizon-size change, with no displacement of another surface at  $2R_H$ . The areal radius is a geometric measure of size, not a universal proper distance. The complete law is

$$dU_\partial = \Pi_\partial dA_H - M\kappa dR_H = T dS - P_R dV_H. \quad (28)$$

Area and volume co-vary along this family:  $\Pi_\partial dA_H = 2P_R dV_H$ ,  $dU_\partial = P_R dV_H$ , and  $U_\partial = 3P_R V_H$ . The mass law remains Eq. (23). Transforming the displacement and its coefficient preserves the selected canonical response of Sec. 3; it adds no independent source variation.

### 5.3. Cosmological application: the FRW screen

For four-dimensional FRW with arbitrary spatial curvature,

$$R_A = \frac{c}{\sqrt{H^2 + kc^2/a^2}}, \quad H = \dot{a}/a. \quad (29)$$

The geometric Misner–Sharp energy at a marginal sphere and its Einstein–FRW matter identification give [15, 22]

$$E_A = \frac{c^4 R_A}{2G} = \rho V_A, \quad V_A = \frac{4\pi R_A^3}{3}. \quad (30)$$

$\rho$  is total bulk energy density and  $p_m$  total bulk pressure in the same units.  $V_A$  is areal volume, not generally the proper spatial volume for  $k \neq 0$ . The relation  $E_A = \rho V_A$  is the established Einstein matter–geometry input, not an independent result of the pressure definition.

The equilibrium FRW (instantaneous Cai–Kim) prescription [14] sets

$$\kappa_{CK} = \frac{c^2}{R_A}, \quad L_{CK} = R_A. \quad (31)$$

Then the force, screen pressure, and integrated identity are

$$F_L^{CK} = \frac{c^4}{2G}, \quad P_{scr} = P_L^{CK} = \frac{c^4}{8\pi G R_A^2} = \frac{\rho}{3}, \quad (32)$$

$$E_A = \rho V_A = U_\partial = TS = F_L L_{CK} = 3P_{scr} V_A. \quad (33)$$

Here  $L_{CK} = R_A$  gives  $F_R = F_L = M_A \kappa_{CK}$  and  $P_R = P_L = P_{scr}$ , with  $M_A = E_A/c^2$ : acceleration-length and radius work coincide. The thermal equality uses the CK temperature and Einstein entropy. The complete constrained differential is

$$dE_A = \Pi_{CK} dA_A - P_{scr} dV_A = T_{CK} dS_A - P_{scr} dV_A. \quad (34)$$

Along the spherical family,  $dE_A/dV_A = P_{\text{scr}}$ . This total slope is not the unconstrained derivative  $-(\partial E/\partial V)_S$  or cosmic material pressure. The evolving apparent horizon is not generally a permanent causal barrier.

Equation (34) is a screen-state law. Cai and Kim's matter-supply relation across a momentarily fixed horizon is  $\delta Q_{\text{matt}} = A_A(\rho + p_m)HR_A dt = T_{\text{CK}}dS_A$  with orientation signs. One sorts change by boundary coordinate; the other sorts it by carrier. Exact mechanical state identities do not by themselves establish exact equilibrium thermality during arbitrary FRW evolution.

With  $\kappa_E = 8\pi G/c^4$ , the pressure representation is

$$\frac{1}{R_A^2} = \frac{H^2}{c^2} + \frac{k}{a^2} = \kappa_E P_{\text{scr}}, \quad (35)$$

$${}^{(2)}\mathcal{R}_\perp \equiv \frac{2\ddot{a}}{ac^2} = -\kappa_E(P_{\text{scr}} + p_m), \quad (36)$$

$$\dot{P}_{\text{scr}} + H(3P_{\text{scr}} + p_m) = 0. \quad (37)$$

$1/R_A^2$  is the horizon sphere's Gaussian curvature;  ${}^{(2)}\mathcal{R}_\perp$  is the Ricci scalar of the normal time-radial metric, not the full spacetime Ricci scalar. Screen pressure sets the horizon scale, its sum with bulk pressure governs acceleration, and bulk enthalpy density  $\rho + p_m = 3P_{\text{scr}} + p_m$  governs its evolution. These give a screen-pressure representation of the same Friedmann dynamics.

The signed Hayward cross-focusing rate is different [16, 17]:

$$\frac{\kappa_H^S}{\kappa_{\text{CK}}} = \frac{p_m - P_{\text{scr}}}{4P_{\text{scr}}} = \frac{3w - 1}{4}, \quad w = p_m/\rho. \quad (38)$$

In radiation FRW it vanishes while  $P_{\text{scr}}$  remains finite; in de Sitter their magnitudes agree. The associated quadratic response is  $P_L^H = (P_{\text{scr}} - p_m)^2/(16P_{\text{scr}})$ . On a nonzero branch its scale work obeys

$$F_L^H dL_H^s = \frac{p_m - P_{\text{scr}}}{4} dV_A - \frac{3}{4} E_A dw. \quad (39)$$

The signed radial coefficient is  $P_R^H = (p_m - P_{\text{scr}})/4$ , distinct from  $P_L^H$  and Hayward work density. Thus  $p_m = P_{\text{scr}} + 4P_R^H$  is an on-shell source reconstruction, not a new equation of state; its geometric and

causal interpretation belongs to the full treatment. Changing rate prescription changes the potential and work allocation.

Kong's effective pressure is  $P_{\text{eff}} = -dE_A/dV_A = -P_{\text{scr}}$  in four-dimensional Einstein FRW [18]. The present normal pressure is obtained before its FRW identification and realized in the stationary sectors above. The distinction concerns its variational origin and mechanical coordinate, not only the sign of a spherical slope.

Ong's shell-volume treatment has  $E = M/2$  and a published spherical pressure alternative [21]. Its thickness-dependent adopted pressure, thermodynamic energy and maximum-force premise differ from the specified boundary-potential response here; see the full comparison.

## 6. Thermal translation and entropy density

A regular Hawking–Unruh/KMS state independently supplies the thermal normalization [7, 19, 20]. Smooth Euclidean normal rotation has period  $2\pi$ , so

$$c\Delta\tau_E = 2\pi L_\kappa, \quad k_B T = \frac{\hbar\Omega_\kappa}{2\pi}, \quad \frac{S}{k_B} = \frac{2\pi J_\eta}{\hbar}. \quad (40)$$

The last identity uses Einstein entropy with the standard zero additive constant. On the positive branch,

$$dU_\partial = T dS - F_L dL_\kappa, \quad \frac{S}{k_B} = \frac{2\pi U_\partial L_\kappa}{\hbar c}. \quad (41)$$

Thus  $U_\partial L_\kappa/c$  has the established meaning of boost charge before its conversion to dimensionless entropy. The area coefficient divided by temperature gives

$$\frac{S}{A} = \frac{P_L L_\kappa}{T} = \frac{k_B}{4\ell_P^2}, \quad \ell_P^2 = \frac{G\hbar}{c^3}. \quad (42)$$

The quarter arises from the Einstein  $1/(8\pi G)$  and thermal  $1/(2\pi)$  normalizations.

## 7. Generality, scope, and conclusion

The contribution is the new length-conjugate boundary force and normal-scale pressure, with

their selected action response and geometric work dictionary. The potential and conjugate rate/area variations are established inputs. The Einstein null sector retains its prescribed polarization, clock, cuts and controlled remaining sources. At  $\kappa = 0$  use the regular rate rather than its reciprocal. Schwarzschild has exact radius work; evolving FRW has a separate instantaneous screen assignment. Neither requires a new interaction; complete moving-boundary matching remains separate.

### Declaration of generative AI and AI-assisted technologies

During preparation and revision, the author used OpenAI ChatGPT and Codex to assist with literature searches, algebraic checks, and presentation. The author takes full responsibility for the manuscript.

### Data availability

No data were created or analyzed in this theoretical study.

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