Horizon mechanics: an explanatory guide

Boundary actions, canonical responses, acceleration length, and horizon scope

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Contents

1 The organizing picture

The papers concern **one gravitational theory, Einstein gravity, examined through several boundary descriptions**. Some formulas are changes of variables within the same description. Others depend on genuinely different boundary conditions, clocks, hypersurfaces, or definitions of energy. These two situations must not be conflated.

The common scalar boundary potential is

$$U_\partial(A,\kappa) = \frac{Ac^2\kappa}{8\pi G}.$$

Here A measures transverse geometry and κ is the acceleration-valued normal rate in a specified clock normalization. On a nonzero signed branch, set

$$L_s = \frac{c^2}{\kappa}, \qquad J_\eta = \frac{Ac^3}{8\pi G}, \qquad \omega_s = \frac{\kappa}{c}.$$

The entire mechanical dictionary is

$$\boxed{U_\partial = \omega_s J_\eta, \qquad U_\partial L_s = cJ_\eta.}$$

$$dU_\partial = \omega_s\,dJ_\eta + J_\eta\,d\omega_s = \Pi_\partial\,dA - F_L\,dL_s.$$

$$\Pi_\partial = \frac{c^2\kappa}{8\pi G}, \qquad F_L = \frac{A\kappa^2}{8\pi G}, \qquad P_L = \frac{F_L}{A} = \frac{\kappa^2}{8\pi G}.$$

These are mechanical statements. No temperature, entropy, Boltzmann constant, or Planck constant is needed. Thermal interpretation is added later. With the positive length $L_\kappa = |L_s|$ and positive rate $\Omega_\kappa = |\kappa|/c$, the corresponding magnitude identities are $|U_\partial| = \Omega_\kappa J_\eta$ and $|U_\partial|L_\kappa = cJ_\eta$. Orientation conventions are those of the papers.

The manuscript's contribution is the physical use of acceleration length as the source coordinate and the response structure it exposes. The area–boost pair, the Einstein area–rate potential, and the thermal normalization have established antecedents. The change of variable is direct once the selected canonical source is supplied. A direct calculation can still have a useful conceptual interpretation; its brevity is not a defect.

Where this comes from: original seven-page paper §§2–4; original full paper Part I and §§II–III. The compact displayed dictionary is an explanatory consolidation. The source archive also contains later two-scale and spherical-completion sections, retained in the revised full manuscript. Source map.

2 From the action to a response

2.1 Action, equations, and boundary terms

An action assigns a number with units of action (energy times time) to a history of fields and a spacetime region. Schematically,

$$I[g,\phi] = I_{\mathrm{EH}}[g] + I_{\mathrm{matter}}[g,\phi] + I_{\mathrm{boundary}} + I_{\mathrm{corner}}.$$

The Einstein–Hilbert part contains spacetime curvature. Matter terms describe the material fields. Boundary and corner terms make the variation appropriate to the data specified on the region's boundary. A corner is a two-dimensional intersection of boundary hypersurfaces, not necessarily a sharp edge in ordinary three-dimensional space.

Varying an action has the schematic structure

$$\delta I = \int_M E \cdot \delta\Phi + \int_{\partial M} \theta(\Phi, \delta\Phi) + \delta I_{\mathrm{boundary/corner}}.$$

Φ collectively denotes the fields; $E = 0$ denotes their equations of motion. The boundary expression depends on variations of the fields and their geometry. This is why the action supplies both bulk equations and boundary response data. A coefficient in the boundary variation does not replace all the bulk equations.

On shell means that the fields satisfy the equations. It does not mean that the action, every boundary contribution, or every variation of the boundary data vanishes. The action is stationary under the variations appropriate to a fixed boundary-value problem. Comparing solutions with different prescribed boundary sources is a different operation.

2.2 Symplectic potential, not potential energy

The integrated boundary variation Θ is called a symplectic potential. It is a **one-form on field space**: give it a small change of the fields and it returns a linear response. Its name should not be confused with the energy-valued potential U_∂.

The ordinary-mechanics prototype is $p\,\delta q$. The coefficient p is conjugate to the varied coordinate q. In field theory there can be one pair at every boundary point, and an integration over the boundary replaces a finite sum. The field-space exterior derivative

$$\Omega = \delta\Theta$$

encodes the canonical pairings. It is a two-form: it compares two independent variations antisymmetrically. Gauge directions can make it degenerate; before those directions are removed, "presymplectic" is the precise term. This Ω is not the boost rate Ω_κ.

The Einstein null-boundary potential has scalar, vector, and trace-free tensor sectors. These are geometric classifications under transformations within a cut. They are not three newly postulated species of matter and do not imply three unconstrained bulk graviton polarizations. [HF17; HF18]

2.3 What is a charge, and when is it a Hamiltonian?

A charge is the quantity associated with a specified transformation. Electric charge generates an internal gauge transformation; an angular momentum generates rotation; a gravitational boost charge generates a change of the normal frame. Its units depend on the parameter being generated. A dimensionless angle naturally has an action-valued conjugate. A time parameter naturally has an energy-valued generator.

In covariant gravity, the Noether current associated with a vector field ξ has the schematic form $\mathcal{J}_\xi = \theta(g, \mathcal{L}_\xi g) - \iota_\xi \mathcal{L}$. On shell it can be written locally as the exterior derivative of a surface-charge form Q_ξ, subject to the usual constraints and conventions. Integrating that form over a cut gives a charge contribution.

A **Hamiltonian** is a generator whose variation is determined by contraction of the symplectic form with the chosen evolution. It exists as a state function only when that variation is integrable with the specified boundaries and flux treatment. A typical adjusted surface expression is

$$\delta H_\xi = \int_S (\delta Q_\xi - \iota_\xi \theta - Q_{\delta\xi}),$$

with additional transport or flux terms when required. The last term matters when the generator itself depends on the fields. Therefore the Noether charge, the complete Hamiltonian, and U_∂ are not synonyms. A boundary potential can be a definite contribution without being the total energy of the spacetime. [IW94; original full paper §III.B]

The same point distinguishes ADM mass at infinity, Brown–York energy conjugate to a boundary clock, and Misner–Sharp energy in spherical symmetry. They answer related but different geometric questions; identical units are not enough to identify them.

3 The null canonical derivation, term by term

3.1 Null hypersurface, cut, and generator

A hypersurface is a three-dimensional subset of four-dimensional spacetime. A null hypersurface has a null normal that is also tangent to it. Its generators are lightlike curves lying within it. A spacelike cut S is a two-dimensional cross-section. The horizon's area refers to such a cut, not to the three-dimensional worldtube swept out by its history.

Let q_{AB} be the positive two-metric on a cut, q its determinant, and

$$\epsilon_S = \sqrt{q}\, d^2x, \qquad A = \int_S \epsilon_S.$$

The capital indices here label the two cut coordinates; the index A is not the total area. A selected generator ξ^a tells us how cuts are followed. Its normalization fixes the parameter along the null generators and must be distinguished from an arbitrary auxiliary null-frame convention.

3.2 Inaffinity and surface gravity

With a generator parameter measured in length units, define

$$\nabla_\xi \xi^a = \hat{\kappa}\, \xi^a, \qquad \kappa = c^2 \hat{\kappa}.$$

The operator $\nabla_\xi = \xi^b \nabla_b$ differentiates along the generator. If its parameter is affine, the left side is zero. A nonzero $\hat{\kappa}$ measures failure of that chosen parameter to be affine: the derivative points along the same null direction rather than perpendicular to it. A null generator has no proper time and no ordinary four-acceleration of a massive observer. Surface gravity is not automatically a local accelerometer reading.

$\hat{\kappa}$ has units of inverse length; κ has units of acceleration. On a stationary Killing horizon, the prescribed Killing normalization supplies the usual surface gravity. For Rindler, observer proper time fixes the normalization so $\kappa = a$. For a Schwarzschild horizon, the standard clock is normalized at infinity. A generic null generator can be reparameterized, which changes its inaffinity; the physical clock prescription is therefore essential.

3.3 Expansion, shear, and the scalar combination

The expansion is the fractional change of area along the null generators:

$$\theta = \frac{1}{\sqrt{q}} \mathcal{L}_\xi \sqrt{q}, \qquad \mathcal{L}_\xi \epsilon_S = \theta \epsilon_S.$$

A small bundle of generators with $\theta > 0$ spreads out; $\theta < 0$ contracts. Shear describes distortion of the cut's shape at fixed infinitesimal area. The scalar sector retains area change and normal parameterization, whereas the trace-free tensor sector retains shape change. With the same length parameter, θ also has units of inverse length.

In four-dimensional Einstein gravity the scalar combination is

$$\mu = \hat{\kappa} + \frac{1}{2}\theta.$$

The one-half is the four-dimensional specialization of $(D-3)/(D-2)$ in the cited decomposition. It comes from the geometric split of the null symplectic potential, not a choice made to obtain P_L. The term “mixed” refers to the combination of intrinsic area data and derivative/extrinsic rate data entering the boundary prescription. It does not mean “mixed quantum state.” [HF17; HF18; ORS23]

3.4 Area versus rate polarization

In this subsection set $c = 1$. The scalar area polarization is

$$\Theta_A^{(0)} = -\frac{1}{8\pi G} \int_{\mathcal{N}} \mu \, \delta \epsilon_S \, dv.$$

The superscript (0) means scalar or spin zero, not perturbative order. The parameter v runs along the generators. This expression treats area as the displayed configuration variable and the negative rate as its conjugate response.

Choose the boundary Legendre term

$$B_\mu = \frac{1}{8\pi G} \int_{\mathcal{N}} \mu \epsilon_S \, dv.$$

Its exact variation changes the potential to

$$\Theta_{\mu}^{(0)} = \Theta_{A}^{(0)} + \delta B_{\mu} = \frac{1}{8\pi G} \int_{\mathcal{N}} \epsilon_S \, \delta\mu \, dv.$$

The elementary identity behind this is $-p\,\delta q + \delta(pq) = q\,\delta p$. Area and rate have exchanged roles as the displayed source. Because $\delta^2 B_{\mu} = 0$, the symplectic two-form is unchanged. This is a genuine **change of polarization**. The subsequent substitution $L_s = c^2/\kappa$ is an invertible source-coordinate change inside the chosen polarization, not another Legendre transform.

These are not two new scalar degrees of freedom. They are different descriptions of the same scalar pairing. Boundary conventions still matter: an additional source-dependent action term can shift a response, and corner terms must be treated consistently. [Original full paper §II.A–B; ORS23]

3.5 Why the force extraction is guaranteed once the canonical term is supplied

Restrict to a nonexpanding null sector and to variations remaining in it:

$$\theta = 0, \qquad \delta\theta = 0.$$

Then $\mu = \hat{\kappa}$. With the prescribed physical time label τ, the scalar response becomes

$$\Theta_{\kappa}^{(0)} = \frac{c^2}{8\pi G} \int d\tau \int_S \epsilon_S \, \delta\kappa.$$

For uniform stationary data and a fixed duration,

$$\frac{\Theta_{\kappa}^{(0)}}{\Delta\tau} = \frac{Ac^2}{8\pi G}\delta\kappa.$$

Yes: this is exactly the normal-rate term in the full differential of the displayed boundary potential. Since

$$\delta\kappa = -\frac{c^2}{L_s^2}\delta L_s,$$

it follows directly that

$$\Theta_{L}^{(0)} = -\int d\tau \int_S \epsilon_S P_L \delta L_s, \qquad P_L = \frac{c^4}{8\pi G L_s^2} = \frac{\kappa^2}{8\pi G}.$$

For a uniform cut, $F_L = AP_L$ and $\Theta_L^{(0)} = -\int F_L \delta L_s \, d\tau$.

There is no second algebraic hurdle. A correctly identified canonical one-form transforms by the chain rule, and its two-form transforms consistently. What is not supplied by the chain rule is the choice of physical boundary, clock, polarization, or allowed source space. Differentiating an arbitrary energy-valued function would not establish those ingredients. The canonical derivation establishes the match for the stated ensemble; it does not certify every other horizon prescription by the same notation.

The complete dU_{∂} is exact; the selected rate leg is not generally exact when A and L_s vary independently. Identifying one leg of dU_{∂} with the source response is not identifying the entire differential with the symplectic potential. [Original full paper §§II.B and VII.A]

3.6 What makes a source variation permitted?

"Permitted" means compatible with the specified variational problem, not independently assumed to be a physical displacement of matter. In this construction the boundary remains null; the variations remain tangent to the chosen nonexpanding sector; cuts and generator normalization are identified by the prescribed clock/reference construction; matter and the other boundary responses are fixed or separately accounted for; and the reciprocal chart stays on a nonzero branch.

There are two different uses of fixed data. Within one conservative boundary-value problem, a prescribed source has $\delta\mu = 0$. To measure the response to that source, one compares nearby problems with different assigned μ while controlling the remaining data. The response is a derivative across that family, just as a susceptibility compares solutions at neighboring applied fields.

A fixed physical clock prescription does not mean $\delta\kappa = 0$. The geometry and its connection can change while the convention defining the time generator remains fixed. Conversely, rescaling the physical clock is not automatically a harmless auxiliary-gauge change.

Not every freely chosen boundary function is guaranteed to admit a compatible solution of the gravitational constraints. The manuscript's variational identity is conditional on the admitted source space; it is not a theorem of existence and uniqueness for every proposed boundary problem. [Original full paper §II.A–B; Appendix C]

3.7 Hamilton–Jacobi meaning and units

In ordinary mechanics the action evaluated on a classical solution is a function of boundary data. Differentiating with respect to an endpoint coordinate yields its momentum; differentiating with respect to a time endpoint yields energy with the appropriate sign. Hamilton–Jacobi theory carries this idea to an action as a function, or functional, of boundary sources.

For the selected mixed gravitational action, its local source response reads

$$\delta I_{\rm os}\big|_{\text{other sources}} = -\int d\tau\, F_L(\tau)\delta L_s(\tau), \qquad F_L(\tau) = -\frac{\delta I_{\rm os}}{\delta L_s(\tau)}.$$

The derivative is functional because $L_s(\tau)$ is a function of time, not merely one number. The integration measure is part of the definition: action divided by time and length has force units. For a time-independent source over a prescribed interval,

$$-\frac{\partial I_{\rm os}}{\partial L_s} = \Delta\tau F_L.$$

This establishes an action response. It does not by itself supply a freely falling material body experiencing that force, nor a protocol for extracting work. The location and sign of the response also depend on the chosen boundary orientation and action terms. [Original full paper Eq. (24)]

4 The corner derivation and the product in the screenshot

4.1 Boost angle and its momentum

The two directions normal to a spacelike cut form a Lorentzian plane. Two choices of normal frame can differ by a Lorentz boost. Its rapidity η is the hyperbolic-angle analogue of an ordinary rotation angle; for two suitable timelike observers in one local plane, relative speed is $v = c\tanh\eta$. At a boundary joint the relevant quantity is the relative boost of the boundary normals. It need not equal a cosmological screen's elementary velocity rapidity.

Einstein's corner structure contains, in the adopted orientation, the one-form

$$\Theta_{\text{corner}} = J_\eta \delta\eta, \qquad J_\eta = \frac{Ac^3}{8\pi G}.$$

This is the same sense of conjugacy as angular momentum times angle. J_η is action-valued and is a gravitational boost charge, not electric charge, rest mass, or ordinary black-hole spin. Its value changes when the area changes; calling it a charge does not imply conservation under every physical process. [CT95; BW12]

4.2 The length momentum explicitly

For a stationary boost flow over a fixed interval,

$$\eta = \frac{\kappa\Delta\tau}{c} = \frac{c\Delta\tau}{L_s}.$$

Consequently

$$p_L \equiv J_\eta \frac{\partial\eta}{\partial L_s} = -\frac{c\Delta\tau J_\eta}{L_s^2} = -\Delta\tau \frac{A\kappa^2}{8\pi G} = -\Delta\tau F_L,$$

and $J_\eta \delta\eta = p_L \delta L_s$.

p_L has ordinary momentum units (force times time); F_L is the response per unit boundary time. They are not the same object and neither is the pressure P_L. The symbols p_L, P_L, p_m, and p_B name different quantities. The stationary interval is essential to this simple integrated expression. For nonstationary boost data the relation is an integral with additional time/geometry variations as required; one cannot insert a changing cosmological rate into this stationary corner identity and call the result a complete moving-screen derivation.

4.3 Why $U_\partial L_s = cJ_\eta$ is revealing

The screenshot's relation follows directly from $U_\partial = \omega_s J_\eta$ and $L_s = c/\omega_s$:

$$\boxed{\frac{U_\partial L_s}{c} = J_\eta = \frac{Ac^3}{8\pi G}.}$$

It says that energy multiplied by its reciprocal-rate time L_s/c is the normal boost charge. At fixed area, changing the clock-normalized rate changes U_∂ and L_s inversely while their product retains the area-normalized action. Under a constant physical-clock rescaling, that product is unchanged. This does not make F_L or P_L separately clock invariant.

The full paper already states $S/k_B = 2\pi U_\partial L_\kappa/(\hbar c)$ on the positive branch (Part I, Eq. S26; main Eq. 84), and both papers state $J_\eta = Ac^3/(8\pi G)$. Their equivalence was not foregrounded as this compact purely mechanical equality. The boost-Hamiltonian interpretation is established literature; the concise acceleration-length packaging is useful exposition of the manuscript's framework, not an independently new entropy law.

The relation uses signed U_∂, L_s, or positive magnitudes $|U_\partial|, L_\kappa$. At exactly $\kappa = 0$, the reciprocal coordinate is undefined; a finite branchwise limit is not permission to multiply zero by infinity at the endpoint.

4.4 Only afterward: the thermal dictionary

Where a regular Hawking–Unruh/KMS interpretation applies,

$$k_B T = \frac{\hbar \Omega_\kappa}{2\pi}, \qquad \frac{S}{k_B} = \frac{2\pi J_\eta}{\hbar}.$$

For positive κ, this gives

$$\omega_s \, dJ_\eta = \Pi_\partial \, dA = T \, dS, \qquad J_\eta \, d\omega_s = -F_L \, dL_\kappa = S \, dT.$$

The word "thermal" belongs here, not to the preceding charge–rate identity. The factor 2π is associated with the regular Euclidean angular period; the gravitational normalization supplies J_η. The quarter-area Einstein entropy follows with its conventional zero additive constant. No planar disk, microscopic tiling, or new thermal postulate has been derived by canceling the fours in $A = 4\pi R^2$.

5 Which horizons are actually covered?

Setting	Direct null/corner certification	Meaning of acceleration length
Rindler, stationary finite patch	Yes, in the stated clock/polarization	Exact observer–horizon proper distance c^2/a
Schwarzschild, stationary horizon	Yes, in the stated clock/polarization	$2R_H$ with time normalized at infinity; not generic proper radial distance
Other stationary Einstein Killing horizons	Same local scalar structure under its assumptions; other charges may be needed	Reciprocal of the selected surface gravity
Exact de Sitter horizon	Stationary null realization available, with consistent clock/orientation	R_A in the conventional static/CK magnitude
Generic FRW apparent-horizon history	The CK state identities hold, but not by direct application of the nonexpanding-null theorem	R_A by the independently specified CK calibration
Spherical dynamical Hayward horizon	Established Kodama–Noether potential and unified balance; not an unrestricted replacement for the null source theorem	Reciprocal cross-focusing rate, generally not R_A

Thus no single restricted nonexpanding-null proof has been established here that directly covers stationary Rindler, Schwarzschild, and every evolving FRW apparent-horizon worldtube with the same physical rate. The algebraic boundary potential supports all three after their stated dictionaries are supplied. The exact FRW application is not thereby invalidated or merely approximate: what differs is its canonical provenance and its thermal-equilibrium status. [Original full paper §VI.C; Cai–Kim]

5.1 A marginal cut is not a nonexpanding null worldtube

At an apparent-horizon cut, one of the two null expansions vanishes. To find the next apparent-horizon cut, however, one follows the locus of marginal spheres, not necessarily that same family of null generators. The worldtube formed by those spheres can be timelike, spacelike, or null. Its area can change even though every constituent sphere is marginal in one null direction.

For flat dust FRW, $a(t) \propto t^{2/3}$ gives $R_A = 3ct/2$. Relative to the comoving flow, the horizon has radial speed $\dot{R}_A - HR_A = c/2$. Its worldtube is timelike and its area increases. The null, nonexpanding hypotheses therefore fail. The velocity rapidity is constant, although the Hayward rate is nonzero. This is why a simple worldtube rapidity derivative does not supply the required horizon boost rate. [Original full paper §VI.G; Faraoni]

There are special null cases, but nullness alone is not enough: the expansion must vanish along the actual generator of the tube. Exact de Sitter supplies the stationary, nonexpanding example. A radiation apparent horizon in flat FRW can be null while the relevant tube expansion is not zero.

5.2 What already holds in FRW

With arbitrary spatial curvature,

$$R_A = \frac{c}{\sqrt{H^2 + kc^2/a^2}}, \qquad \kappa_{\rm CK} = \frac{c^2}{R_A}, \qquad L_{\rm CK} = R_A.$$

Using the Einstein–FRW matter relation $E_{\rm MS} = \rho V_A = c^4 R_A/(2G)$ and areal volume $V_A = 4\pi R_A^3/3$,

$$P_{\rm scr} = \frac{\rho}{3} = P_L^{\rm CK}, \qquad E_{\rm MS} = U_\partial = F_L L_{\rm CK} = 3P_L^{\rm CK} V_A.$$

The complete mechanical state differential is

$$dE_{\rm MS} = \Pi_{\rm CK}\, dA_A - P_{\rm scr}\, dV_A.$$

The thermal version uses $\Pi_{\rm CK} dA_A = T_{\rm CK} dS_A$. This is a constrained screen-state law, not the matter energy-flux law across a momentarily fixed horizon. The bulk-energy identification is explicit gravitational input, not a second density invented for presentation. No new bulk Friedmann dynamics are claimed.

6 Can the canonical argument be extended?

6.1 What can already be said on an expanding null boundary

There is a precise partial extension available from the same scalar one-form. Define $\mu_{\rm acc} = c^2\mu = \kappa + c^2\theta/2$. With the same fixed clock-measure and polarization conventions, its scalar part is

$$\Theta_\mu^{(0)} = \int d\tau \int_S \epsilon_S \left(\frac{c^2}{8\pi G}\delta\kappa + \frac{c^4}{16\pi G}\delta\theta \right).$$

Using $L_s = c^2/\kappa$ gives the exact scalar coordinate identity

$$\boxed{\Theta_\mu^{(0)} = -\int d\tau \int_S \epsilon_S P_L \delta L_s + \int d\tau \int_S \epsilon_S \frac{c^4}{16\pi G}\delta\theta.}$$

Therefore the familiar surface-gravity response survives as a **partial scalar coefficient**; an independent expansion response remains. Setting $\delta\theta = 0$ removes that displayed term, even at a nonzero fixed expansion, provided the remaining boundary conditions really admit the chosen variations. This last proviso is a geometric/source-space question, not answered by algebra alone.

Alternatively, on $\mu \neq 0$ one can use $L_\mu = 1/\mu$ and write the whole scalar sector as $-\int \epsilon_S P_\mu \delta L_\mu \, d\tau$ with $P_\mu = c^4/(8\pi G L_\mu^2)$. But L_μ is generally **not** c^2/κ. Neither version removes shear, generator, matter, clock, or corner responses. This is an algebraic consequence of the established scalar potential, added explicitly in this explanatory revision; it is not promoted to a complete dynamical horizon theorem.

6.2 What a full moving-boundary theorem would have to establish

"Generic moving null-boundary theorem" is not a particular standard theorem name. Two problems should be distinguished: an evolving but still null hypersurface with expansion/shear, and an apparent-horizon worldtube whose causal type may itself be non-null. The latter requires an appropriate timelike/spacelike or signature-aware treatment, not merely a null formula with time-dependent symbols.

A complete theorem for a selected horizon prescription would specify its geometry, embedding, physical evolution, matter boundary data, polarization, counterterms, and corner conditions. It would calculate the full on-shell boundary variation, then demonstrate which response is paired with the original intended L_{CK} or L_H, with all remaining source and transport terms retained. It must also distinguish a local response coefficient from an integrable Hamiltonian charge.

The papers already establish several pieces, including the general null scalar structure, the stationary boost check, spherical Noether identities, and the explicit timelike calculation. What is not established is their universal identification with the CK or Hayward rate on arbitrary moving FRW screens. The timelike counterexample below disproves that identification for **that ensemble**, not for all ensembles. The stretched-Carrollian formalism is relevant because it retains clock, stretching, and other responses in a common description. [Full paper Appendix C; Freidel–Jai-akson §5.1]

7 Why ζ_B is defined that way

Use $c = 1$, as in Appendix C. A spherical timelike worldtube has induced metric $-N_B^2 dt^2 + R_B^2 d\Omega_2^2$. The lapse N_B relates proper boundary time to the coordinate time. The on-shell Dirichlet boundary response is

$$\delta I_D\big|_B = \int dt \left[-E_B \delta N_B + N_B p_B \delta A_B\right].$$

E_B is the unreferenced Brown–York energy conjugate to the clock variable; p_B is the isotropic two-dimensional surface stress, not bulk pressure. To prescribe stress rather than area, choose

$$I_M = I_D - \int dt \, N_B A_B p_B.$$

The result is

$$\delta I_M\big|_B = \int dt\,[-E_B\delta N_B - A_B\delta(N_B p_B)]\,.$$

Only now introduce the notation

$$\zeta_B \equiv -8\pi G N_B p_B,$$

which puts the second term into $A_B\delta\zeta_B/(8\pi G)$. The factor $8\pi G$ removes the conventional Einstein normalization; the lapse places the local surface stress in the selected clock convention; the minus sign matches the chosen orientation/polarization. **It is a normalization choice, not a derivation that this source equals surface gravity.**

The geometry gives its independent content. With outward normal n^a, boundary unit timelike vector u^a, and the paper's extrinsic-curvature convention,

$$a_n = n_a u^b \nabla_b u^a, \qquad b_B = \frac{n^a\nabla_a R_B}{R_B}, \qquad p_B = \frac{a_n + b_B}{8\pi G},$$

so

$$\boxed{\zeta_B = -N_B(a_n + b_B).}$$

a_n is the normal component of the observers' acceleration; b_B measures the normal change of the transverse radius per radius. In spherical symmetry it is the angular extrinsic-curvature eigenvalue. The broader quantity is therefore a **clock-weighted combination of normal acceleration and transverse extrinsic curvature**, namely the surface-stress source of this boundary problem.

On $\zeta_B \neq 0$, the reciprocal coordinate $L_B = 1/\zeta_B$ gives

$$\delta I_M\big|_B = \int dt[-E_B\delta N_B - F_B\delta L_B], \qquad F_B = \frac{A_B\zeta_B^2}{8\pi G}.$$

This variational calculation is meaningful within the specified action and matter/endpoint conditions. Its square-law shape also reflects using a reciprocal coordinate. An auxiliary $U_B = A_B\zeta_B/(8\pi G)$ is not generally E_B or the Misner–Sharp energy. The action's force-like response becomes the paper's original horizon response only after $\zeta_B = \kappa/c^2$ is independently established with matching normalization.

In flat dust FRW the appendix finds $\zeta_B = -2/R_A$, whereas $\kappa_{\rm CK} = 1/R_A$ and $\kappa_H^s = -1/(4R_A)$ in these units. The mismatch is real and cannot be removed merely by flipping orientation. In stationary stretched Schwarzschild and de Sitter limits, a matching horizon rate is recovered. Thus this appendix is best read as a **scope test of the larger extrinsic boundary response**, not as another proof of the CK formula. [Original full paper Appendix C]

8 How the pressure enters membrane mechanics

The surface stress Π_∂ has units of energy per area or force per length. The pressure P_L has force-per-area units. They are linked by

$$\Pi_\partial(L_s) = \frac{c^4}{8\pi G L_s} = P_L L_s, \qquad d\Pi_\partial = -P_L dL_s.$$

On a nonuniform cut, let $L_s = L_s(x)$. The tangential derivative D_i is the covariant spatial gradient using the cut metric. Pulling back the source-space identity means evaluating it on this spatially varying field:

$$D_i\Pi_\partial = -P_L D_i L_s, \qquad -D_i\Pi_\partial = P_L D_i L_s.$$

The last expression is the familiar surface-pressure-gradient contribution to tangential membrane momentum balance, expressed through the acceleration-length gradient. There is no new force law hidden in the word "pullback"; it is the geometric chain rule. The physical content is the connection to the already established membrane stress.

Normal balance is a different projection. In the matched static timelike-screen convention of the paper,

$$0 = \Delta p + \gamma\mathscr{H} = \Delta p - P_L L_s \mathscr{H},$$

where $\gamma = -\Pi_\partial$ is oriented surface tension, $\mathscr{H}$ is the trace of the cut's spatial extrinsic curvature (the sum of principal curvatures), and Δp is the appropriate oriented normal-pressure difference including the screen equation's curvature contribution. $\mathscr{H}$ is not the cosmological Hubble rate H. On a round sphere in Euclidean three-space this trace is $2/R$.

Thus the corresponding traction is $\Delta p = P_L L_s \mathscr{H}$, not universally P_L by itself. The dimensionless factor $L_s\mathscr{H}$ converts the normal-scale response to curvature pressure. The full dynamical screen equation also contains expansion, shear, normal-connection, and lapse-gradient contributions. Those do not disappear by substituting $\Pi_\partial = P_L L_s$. A screen normal rate must be matched to the desired horizon prescription rather than identified by its symbol alone. [Original full paper §IV.C–D; Freidel–Yokokura]

9 FGP distance and acceleration length

Frodden, Ghosh, and Perez select stationary observers at a small proper distance ℓ outside a nonextremal horizon. At leading near-horizon order,

$$E_{\mathrm{FGP}} \simeq \frac{Ac^4}{8\pi G\ell}, \qquad \kappa_{\mathrm{loc}} \simeq \frac{c^2}{\ell}, \qquad L_{\mathrm{loc}} = \frac{c^2}{\kappa_{\mathrm{loc}}} \simeq \ell.$$

At fixed area, the negative distance derivative is

$$-\left(\frac{\partial E_{\mathrm{FGP}}}{\partial \ell}\right)_A \simeq \frac{Ac^4}{8\pi G\ell^2}.$$

This is the "inverse-square coefficient": doubling the distance divides the leading coefficient by four. Dividing by area gives the same local quadratic-rate form, to that order. It is a close antecedent, not an algebraically unrelated energy formula.

The differences concern what the variables mean and what is varied. FGP's first law compares nearby horizon states while maintaining the observer-distance prescription, giving the area term. Taking a derivative in ℓ instead moves among observer/clock choices as well. The existence of that derivative does not alone establish the complete action work for moving the observer or its equality to a force on a material body. The null derivation specifies a gravitational rate-source problem and obtains its action coefficient there.

For Schwarzschild, $N(r) = \sqrt{1 - R_H/r}$ gives

$$\kappa_{\text{loc}} = \frac{\kappa_\infty}{N(r)}, \qquad L_{\text{loc}} = N(r)L_\infty = 2R_H N(r), \qquad L_\infty = 2R_H.$$

Near the horizon, $\ell \simeq 2\sqrt{R_H(r - R_H)}$ and $N \simeq \sqrt{(r - R_H)/R_H}$, so $L_{\text{loc}} \simeq \ell$. Farther away this equality fails. The redshifted horizon surface gravity is also not exactly the local observer's proper acceleration away from the horizon. Rindler is the setting where the proper-distance equality is exact.

The two frameworks overlap locally when the observer and normalization are matched. The manuscript's L_s is a reciprocal selected rate applicable across its stated sectors; FGP's ℓ is a directly specified local proper distance. Neither the smallness of ℓ nor the finiteness of L_s alone decides whether a legitimate work coordinate exists. [Original full paper §V.A; FGP]

10 Why Hayward work is not CK work

The geometric prescriptions differ:

$$\kappa_H^s = -\frac{c^2}{R_A}\left(1 - \frac{\dot{R}_A}{2HR_A}\right), \qquad \kappa_{\text{CK}} = \frac{c^2}{R_A}.$$

Hayward's rate probes normal cross-focusing and depends on evolution; CK supplies the instantaneous equilibrium radius scale. In Einstein–FRW, with $w = p_m/\rho$ and $\chi = \kappa_H^s/\kappa_{\text{CK}} = (3w - 1)/4$,

$$\widetilde{U}_\partial = \chi E_A, \qquad L_H^s = \frac{R_A}{\chi}, \qquad P_L^H = \chi^2 P_{\text{scr}}.$$

Direct differentiation gives

$$\boxed{F_L^H\, dL_H^s = \chi P_{\text{scr}}\, dV_A - E_A\, d\chi.}$$

The extra term is essential when the equation of state changes. Even at constant w, the coefficient differs from CK's unless the signed or magnitude prescriptions coincide appropriately. The original full paper already derives this exact relationship; it is not merely an unexplained warning that the two work terms differ.

Hayward's matter work density is $W = (\rho - p_m)/2$. The projected unified first law is $d_H E_A = T_H^s d_H S_A + W d_H V_A$, while the CK screen-state law is $dE_A = T_{\text{CK}} dS_A - P_{\text{scr}} dV_A$. Different area coefficients compensate the different work allocations on the physical horizon trajectory. They are compatible balances, not identical terms. In radiation FRW the Hayward rate vanishes while CK pressure remains finite, making a universal identification impossible. Exact de Sitter is a special stationary agreement in magnitudes. [Original full paper §VI.E–F]

11 What the revised presentations do

The one-page presentation keeps the direct scalar substitution, the purely mechanical boost-product dictionary, the FRW energy identification, and the pressure-form cosmological equations. It preserves the author's requested omission of name/location and the intermediate $P_L A R_A$ term.

The seven-page presentation explains the canonical terminology briefly at first use, displays $U_\partial L_s = cJ_\eta$, separates mechanical and thermal dictionaries, and states the horizon scope next to the canonical argument. It does not promote the timelike stress comparison to a principal horizon theorem.

The full working version retains the detailed source-archive material and adds the action-to-response hierarchy, the expanded null-scalar identity, the clock/charge interpretation, and sharper explanations of FGP, membrane balance, and the Appendix C normalization. The moving-screen identification remains a stated target rather than an asserted completed theorem. The change log distinguishes retained content from new explanatory observations.

12 Source map and reading links

The glossary pages use stable internal anchors and give source locations in the original manuscripts. Original section numbers are retained in the descriptions even when revised pagination changes. The package includes reference copies of the supplied seven- and 41-page PDFs, the revised LaTeX sources, and separate revision notes.

- **Original seven-page manuscript:** *Horizon Pressure from the Einstein Boundary Potential*, supplied PDF. §§2–3: mechanical and canonical derivation; §3.1: boost check; §4: mass-law distinction; §5: horizon realizations; §6: thermality; §7: scope. Reference PDF.
- **Original full manuscript:** *Surface Gravity Squared as Gravitational Boundary Pressure*, supplied 41-page PDF. §§II–III: source and clock; §IV: membrane and thermal structure; §V: stationary sectors/FGP; §VI: cosmology; §VII: integrability/dynamics; §VIII: Wald extension; Appendix C: timelike comparison. Reference PDF.
- **HF17:** Hopfmüller–Freidel, *Gravity degrees of freedom on a null surface.* arXiv:1611.03096.
- **HF18:** Hopfmüller–Freidel, *Null conservation laws for gravity.* arXiv:1802.06135.
- **ORS23:** Odak–Rignon-Bret–Speziale, *General gravitational charges on null hypersurfaces.* arXiv:2309.03854.
- **CT95:** Carlip–Teitelboim, *The off-shell black hole.* arXiv:gr-qc/9312002.
- **BW12:** Bianchi–Wieland, *Horizon energy as the boost boundary term in general relativity and loop gravity.* arXiv:1205.5325.
- **IW94:** Iyer–Wald, *Some properties of Noether charge and a proposal for dynamical black hole entropy.* arXiv:gr-qc/9403028.
- **Harlow–Wu:** *Covariant phase space with boundaries.* arXiv:1906.08616.
- **FGP:** Frodden–Ghosh–Perez, *Quasilocal first law for black hole thermodynamics.* arXiv:1110.4055.
- **Cai–Kim:** *First law of thermodynamics and Friedmann equations of FRW universe.* arXiv:hep-th/0501055.
- **Faraoni:** *Cosmological apparent and trapping horizons.* arXiv:1106.4427.
- **Freidel–Yokokura:** *Non-equilibrium thermodynamics of gravitational screens.* arXiv:1405.4881.
- **Freidel–Jai-akson:** *Geometry of Carrollian stretched horizons*, §5.1. arXiv:2406.06709.

These sources establish the antecedent geometric/canonical structures. The acceleration-length interpretations and their stated restrictions are the manuscript's synthesis. Algebraic tests in the package check the displayed transformations; they are not independent peer review or a proof of an arbitrary moving-boundary extension.