Surface Gravity Squared as Gravitational Boundary Pressure: Acceleration-Length Response, Normal Force Balance, and FRW Screen Work

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On a clock-anchored, non-expanding null boundary, the established Einstein normal-rate source admits the branchwise acceleration length $L_{\mathrm{s}} = c^2/\kappa$. Its Hamilton–Jacobi response is the normal pressure

$$P_L = \frac{\kappa^2}{8\pi G}, \qquad F_L = AP_L,$$

and the boundary term is $-P_L\,\delta V_\perp$ at fixed transverse area. Thus the normally hidden surface-gravity variation has the same force-times-displacement and pressure-times-volume form as ordinary mechanical work. The area–boost corner pair independently yields the same force. The reciprocal length is simultaneously a mechanical and thermal scale, $L_\kappa = c^2/|\kappa| = \hbar c/(2\pi k_B T)$, so Einstein boundary mechanics and Hawking–Unruh temperature give the paired laws $P_L \propto L_\kappa^{-2}$ and $T \propto L_\kappa^{-1}$ and reconstruct the quarter-area entropy density without assuming conventional state counting.

The construction also clarifies how the new normal pressure is related to, but distinct from, the established two-dimensional membrane pressure $\Pi_\partial = c^2\kappa/(8\pi G)$. On a signed branch, $\Pi_\partial = P_L L_{\mathrm{s}}$ and $D_A\Pi_\partial = -P_L D_A L_{\mathrm{s}}$; equivalently, $|\Pi_\partial| = P_L L_\kappa$ and $D_A|\Pi_\partial| = -P_L D_A L_\kappa$. Thus the membrane pressure-gradient one-form is the pullback to the horizon cut of the source-space response $-P_L dL_{\mathrm{s}}$. In the positive-κ orientation, the gravitational-screen Young–Laplace equation gives the curvature pressure $P_L L_\kappa H$.

Rindler realizes L_κ as proper observer–horizon distance. Schwarzschild gives an exact pointwise thermal–mechanical virial duality, $Mc^2 = TS + F_L L_\kappa$ with $TS = F_L L_\kappa = Mc^2/2$; the mechanical product equals the on-shell Helmholtz free energy. Cai–Kim FRW gives $P_L = P_{\mathrm{scr}} = \rho/3$ and $dE_A = T_{\mathrm{CK}} dS_A - P_{\mathrm{scr}} dV_A$. An exact dynamical-horizon theorem turns the nonrotating scale leg into a finite constant-force flux law, while a phase-space curl identifies precisely when it is not a separate state energy. A stationary Wald-entropy extension and new response functions are also given. The result is an action-supported mechanical representation of established gravitational boundary structure, not a new bulk stress tensor or field equation.

I. INTRODUCTION

Horizon mechanics naturally singles out area and surface gravity. Horizon thermodynamics repackages the same pair as entropy and temperature. What is less often made explicit is the mechanical content of varying the normal rate itself. In a stationary first law the horizon generator and its clock normalization are normally held fixed, so the area variation is visible while the complementary $A\,\delta\kappa$ direction is absent or absorbed by the definition of the charge. This is appropriate for that ensemble, but it leaves open a different boundary-value question: if the prescribed normal rate is allowed to vary as a source, what is its response?

The reciprocal length

$$L_{\mathrm{s}} = \frac{c^2}{\kappa} \tag{1}$$

turns that question into a mechanical one. It is the light-travel length associated with the inverse surface-gravity timescale. In Rindler spacetime it is exactly the proper distance from a uniformly accelerated observer to that observer's horizon. For a stationary black hole it is an inverse redshift-normalized surface-gravity scale. In spherical dynamics its Hayward analogue is an inverse normal cross-focusing scale.

The common Einstein boundary potential used below is

$$U_\partial(A, L_{\mathrm{s}}) = \mathcal{T}_G \frac{A}{L_{\mathrm{s}}}, \qquad \mathcal{T}_G \equiv \frac{c^4}{8\pi G}, \tag{2}$$

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whose fixed-area derivative gives

$$F_L = -\left(\frac{\partial U_\partial}{\partial L_{\rm s}}\right)_A, \qquad P_L \equiv \frac{F_L}{A} = \frac{c^4}{8\pi G L_{\rm s}^2} = \frac{\kappa^2}{8\pi G}. \tag{3}$$

The main purpose of the paper is not to infer Eq. (3) from dimensional analogy or from TS. It is to derive it directly from the gravitational boundary variation, identify the clock and gauge conditions under which it is meaningful, and determine what it does in stationary, cosmological, and dynamical sectors.

The generic scalar source on a null boundary is not surface gravity alone but

$$\mu = \widehat{\kappa} + \frac{1}{2}\theta, \tag{4}$$

where $\widehat{\kappa} = \kappa/c^2$ is the inverse-length inaffinity and θ is the null expansion. The clean $\kappa \leftrightarrow L_{\rm s}$ source chart therefore applies canonically on a clock-anchored non-expanding sector, not on every moving horizon. Other sectors can support the same algebra only after an independently established prescription for their normal rate and parent $A\kappa$ potential has been supplied.

The paper develops four additional consequences of the source response. First, the signed and magnitude relations

$$\Pi_\partial = P_L L_{\rm s}, \qquad |\Pi_\partial| = P_L L_\kappa, \qquad \Pi_\partial \equiv \frac{c^2\kappa}{8\pi G}, \tag{5}$$

link the new normal pressure to the established two-dimensional membrane pressure. The link is stronger than a comparison of units: for a nonuniform signed source on a cut, $D_A \Pi_\partial = -P_L D_A L_{\rm s}$; equivalently, on a fixed-sign branch, $D_A|\Pi_\partial| = -P_L D_A L_\kappa$. This is the coordinate-free pullback of the source-space response one-form to the physical cut and turns the usual membrane pressure gradient into a spatial manifestation of the normal scale response.

Second, the same substitution converts the gravitational-screen Young–Laplace equation into a normal force balance. With the usual orientation in which the surface tension is $\gamma = -\Pi_\partial$, the exact signed equation and its positive-κ form are

$$\Delta p - \Pi_\partial H = \Delta p - P_L L_{\rm s} H = 0, \qquad \Delta p - P_L L_\kappa H = 0 \quad (\kappa > 0). \tag{6}$$

Here H is the mean curvature and Δp is the appropriate outside-minus-inside normal pressure. Equation (6) is not a new Einstein equation; it is an acceleration-length rewriting of an established normal projection of Einstein's equation. It does, however, answer directly the question of whether P_L can enter a genuine force-balance law.

Third, L_κ is a mechanical–thermal bridge scale:

$$L_\kappa = \frac{c^2}{|\kappa|} = \frac{\hbar c}{2\pi k_B T}. \tag{7}$$

Classical boundary mechanics assigns it the inverse-square law $P_L(L_\kappa) = c^4/(8\pi G L_\kappa^2)$, while quantum field theory assigns it the inverse-linear law $k_B T(L_\kappa) = \hbar c/(2\pi L_\kappa)$. Their ratio fixes the Planck-area entropy density.

Fourth, Schwarzschild displays an exact pointwise thermal–mechanical Euler/virial split rather than a relation obtained only after time averaging:

$$Mc^2 = TS + F_L L_\kappa, \qquad TS = F_L L_\kappa = \frac{Mc^2}{2}. \tag{8}$$

The two products are distinct operational channels–area/entropy and scale/mechanical response–although they are constrained equal on the one-parameter Schwarzschild family. Along that family they co-vary exactly, $d(TS) = d(F_L L_\kappa) = \frac{1}{2}d(Mc^2)$: adding mass increases both halves in lockstep. That is ordinary virial-family behavior, not a failure of the analogy. The formal on-shell Helmholtz free energy is $Mc^2 - TS = F_L L_\kappa$. This supports, but does not by itself prove, a kinetic-like versus potential/free-energy interpretation.

The novelty is not the algebraic chain rule $\kappa \mapsto c^2/\kappa$, nor the underlying $A\kappa$ potential. It is the identification of acceleration length as an admissible and mechanically privileged mixed-boundary source coordinate; the associated action response and clock, gauge, and integrability conditions; the normal and tangential force-balance relations it organizes; and the consequences across Rindler, stationary black-hole, cosmological, and dynamical sectors.

Sections II–IV establish the source, gauge, canonical, mechanical, and thermal structure. Section V treats stationary realizations and the thermal–mechanical virial duality. Section VI develops the FRW screen law and Hayward focusing response. Section VII treats integrability and finite dynamical balance. Section VIII gives the Wald-entropy extension and Section IX states the contribution and its limits.

II. NON-EXPANDING NULL-BOUNDARY SOURCE-RESPONSE PROPOSITION

A. Established null scalar pair

Let $\mathcal{N}$ be a null hypersurface foliated by spacelike cuts S. Let ξ^a be a chosen null evolution vector tangent to $\mathcal{N}$, and let q_{AB} and $\epsilon_S = \sqrt{q}\, d^2x$ be the induced metric and area form on a cut. Define the inaffinity and expansion by

$$\nabla_\xi \xi^a = \widehat{\kappa}_\xi \xi^a, \qquad \theta_\xi = \frac{1}{\sqrt{q}} \mathcal{L}_\xi \sqrt{q}. \tag{9}$$

For the canonical formulas in this subsection set $c = 1$, take the null parameter v to have units of length, and retain $8\pi G$ explicitly. Hopfmüller and Freidel derived the null symplectic potential without initially fixing the diffeomorphism gauge. In four dimensions its bulk part has the schematic spin decomposition [7]

$$\Theta_{\mathcal{N}}^{\text{bulk}} = \frac{1}{8\pi G} \int_{\mathcal{N}} \left[\frac{1}{2} \widetilde{\sigma}^{AB} \delta \widetilde{q}_{AB} - \overline{\eta}_A \delta \xi^A - \left(\widehat{\kappa}_\xi + \frac{1}{2}\theta_\xi \right) \delta \ln \sqrt{q} \right] \epsilon_{\mathcal{N}}. \tag{10}$$

The first two terms are the spin-two and spin-one sectors. The spin-zero momentum is

$$\boxed{\mu_\xi = \widehat{\kappa}_\xi + \frac{1}{2}\theta_\xi.} \tag{11}$$

Writing $\epsilon_{\mathcal{N}} = dv \wedge \epsilon_S$, the scalar area polarization is

$$\Theta_A^{(0)} = -\frac{1}{8\pi G} \int_{\mathcal{N}} \mu_\xi \, \delta \epsilon_S \, dv. \tag{12}$$

Adding the exact variation of

$$B_\mu = \frac{1}{8\pi G} \int_{\mathcal{N}} \mu_\xi \epsilon_S \, dv \tag{13}$$

changes the scalar polarization without changing the symplectic two-form:

$$\Theta_\mu^{(0)} = \Theta_A^{(0)} + \delta B_\mu = \frac{1}{8\pi G} \int_{\mathcal{N}} \epsilon_S \, \delta \mu_\xi \, dv. \tag{14}$$

Hopfmüller and Freidel interpret μ as a gravitational boundary pressure and argue that fixing it is a natural clock boundary condition for null translations [8]. This established pressure is linear in μ and is not the quadratic force-per-area response introduced below.

Odak, Rignon-Bret, and Speziale subsequently obtained the covariant conformal/York polarization for general null variations [9]. In their conventions the spin-zero source is $\theta + 2\widehat{\kappa} = 2\mu$. Their conservative mixed conditions fix that quantity together with appropriate spin-two, generator, and corner data. The scalar condition is Robin-type because it contains metric derivatives. Thus the use of μ as a mixed-boundary source is an established variational statement rather than an assumption of the present construction.

B. Acceleration-length source representation

Restrict to a phase-space sector consisting of non-expanding null boundaries and variations tangent to that sector:

$$\theta_\xi = 0, \qquad \delta\theta_\xi = 0. \tag{15}$$

Then $\mu_\xi = \widehat{\kappa}_\xi$. Restore physical units by defining the acceleration-valued surface gravity

$$\kappa_\xi \equiv c^2 \widehat{\kappa}_\xi. \tag{16}$$

Expressing the null evolution in a prescribed physical clock τ gives Eq. (14) in the form

$$\Theta_\kappa^{(0)} = \frac{c^2}{8\pi G} \int d\tau \int_S \epsilon_S \, \delta \kappa_\xi. \tag{17}$$

On either nonzero signed branch define

$$L_{\rm s} \equiv \frac{c^2}{\kappa_\xi}, \qquad \delta\kappa_\xi = -\frac{c^2}{L_{\rm s}^2}\delta L_{\rm s}. \tag{18}$$

Substitution yields

$$\boxed{\Theta_L^{(0)} = -\int d\tau \int_S \epsilon_S \, \frac{c^4}{8\pi G L_{\rm s}^2}\, \delta L_{\rm s}.} \tag{19}$$

The local response density is therefore

$$\boxed{P_L = \frac{c^4}{8\pi G L_{\rm s}^2} = \frac{\kappa_\xi^2}{8\pi G}.} \tag{20}$$

For a source uniform on a finite cut,

$$F_L \equiv \int_S P_L \, dA = A P_L, \qquad \Theta_L^{(0)} = -\int d\tau \, F_L \delta L_{\rm s}. \tag{21}$$

At fixed transverse area define the swept normal-volume displacement

$$\delta V_\perp \equiv A \, \delta L_{\rm s}. \tag{22}$$

Then

$$\boxed{\Theta_L^{(0)} = -\int d\tau \, P_L \delta V_\perp.} \tag{23}$$

This is the direct mechanical reason for the word pressure: P_L is force per transverse area and multiplies a normal volume displacement. The terminology does not rest on $U_\partial = TS$ or on dimensional resemblance alone.

For the selected mixed action, orientation, cuts, counterterm prescription, and fixed remaining sources, the local-in-time Hamilton–Jacobi relation is

$$F_L(\tau) = -\frac{\delta I_{\rm os}}{\delta L_{\rm s}(\tau)}. \tag{24}$$

If the source is constant through a stationary interval $\Delta\tau$, then $-(\partial I_{\rm os}/\partial L_{\rm s})_{\Delta\tau} = \Delta\tau F_L$. Here canonical response means the coefficient paired with the chosen source in the Einstein symplectic potential, equivalently the functional derivative of the mixed on-shell action. It is not a claim that P_L is a covariant local bulk stress tensor.

The corresponding extended-source two-form is

$$\Omega_L^{(0)} = \delta\Theta_L^{(0)} = \int d\tau \int_S \delta L_{\rm s} \wedge \delta(\epsilon_S P_L). \tag{25}$$

The boundary Legendre transform from Eq. (12) to Eq. (14) is the genuine change of polarization. The subsequent map $\kappa \mapsto c^2/\kappa$ is an invertible source coordinate change inside the rate-source polarization, not a second Legendre transform.

Any smooth invertible coordinate $q = f(\kappa)$ would transform the response by the chain rule. The action does not select $L_{\rm s}$ by algebra alone. Acceleration length is mechanically privileged because, up to a dimensionless factor, it is the unique length constructible from c and κ without another scale; Rindler proper distance fixes that factor to one. It is also the reduced inverse-temperature scale, the coordinate in which the response has pressure units, and the coordinate for which $F_L L_{\rm s} = U_\partial$. In Cai–Kim FRW it becomes the areal radius and converts exactly to spherical $P\,dV$ work.

The result may be summarized as follows.

> *Non-expanding null-boundary source-response proposition.* Let $\mathcal{N}$ be a four-dimensional non-expanding Einstein null boundary equipped with a prescribed physical evolution vector, fixed relational cuts, and variations tangent to the non-expanding sector. Assume that matter boundary sources and omitted symplectic flux are controlled separately. On either branch $\kappa_\xi \neq 0$, the signed acceleration length $L_{\rm s} = c^2/\kappa_\xi = 1/\widehat{\kappa}_\xi$ is an admissible reparameterization of the covariant mixed spin-zero source. Its Hamilton–Jacobi response density is $P_L = \kappa_\xi^2/(8\pi G)$, and a uniform finite cut carries $F_L = A P_L$.

The hypotheses are substantive. On a generic expanding null boundary the source is μ, so after defining $\mu_{\rm acc}=c^2\mu$ the clean reciprocal coordinate is

$$L_\mu=\frac{c^2}{\mu_{\rm acc}}, \tag{26}$$

not c^2/κ alone. At $\kappa=0$ the reciprocal chart is singular, while the parent rate is regular. Indeed,

$$-P_L\delta L_{\rm s}=-\frac{\kappa_\xi^2}{8\pi G}\delta\left(\frac{c^2}{\kappa_\xi}\right)=\frac{c^2}{8\pi G}\delta\kappa_\xi. \tag{27}$$

Thus $P_L\to 0$ at $\kappa\to 0$ does not make the parent variation vanish; it means that the L chart has reached infinity.

C. Independent area–boost derivation

The stationary corner sector supplies an independent check. Einstein gravity assigns the relative boost angle η of the boundary normals the corner momentum

$$J_\eta=\frac{c^3A}{8\pi G} \tag{28}$$

[11–13]. For a uniform stationary interval of prescribed duration $\Delta\tau$,

$$\eta=\frac{\kappa_\xi\Delta\tau}{c}=\frac{c\Delta\tau}{L_{\rm s}}. \tag{29}$$

At fixed A and $\Delta\tau$,

$$J_\eta\delta\eta=-\Delta\tau F_L\delta L_{\rm s},\qquad \Omega_{\rm corner}=\Delta\tau\,\delta L_{\rm s}\wedge\delta F_L. \tag{30}$$

The momentum conjugate to $L_{\rm s}$ in this stationary edge-mode sector is $p_L=-\Delta\tau F_L$; F_L is the response per unit boundary time. A force appears because the boost angle is a rate integrated over time and $L_{\rm s}$ is the reciprocal of that rate.

Equations (17) and (30) provide two independent routes to the same coefficient. Therefore the central canonical algebra is not contingent on a thermodynamic analogy. Its remaining conditionality concerns the selected boundary ensemble, clock, cuts, and allowed counterterms, not the chain-rule calculation itself.

III. CLOCK, GAUGE, AND THE STANDARD FIRST LAW

A. Auxiliary normal versus physical clock

Surface gravity is a rate relative to a normal evolution. To separate an auxiliary null-frame choice from a physical clock, write

$$\xi^a=f\ell^a,\qquad \widehat{\kappa}_\xi=f\widehat{\kappa}_\ell+\ell[f]. \tag{31}$$

Under the simultaneous change

$$\ell^a\longrightarrow e^\sigma\ell^a,\qquad f\longrightarrow e^{-\sigma}f, \tag{32}$$

the physical vector ξ^a is unchanged, and direct substitution shows that $\widehat{\kappa}_\xi$, κ_ξ, and $L_{\rm s}$ are unchanged. The construction is therefore independent of the auxiliary null representative for a fixed physical normal flow [10].

Changing the physical clock is different. If $\xi^a\to\alpha\xi^a$, then

$$\widehat{\kappa}_{\alpha\xi}=\alpha\widehat{\kappa}_\xi+\xi[\alpha],\qquad \kappa_{\alpha\xi}=\alpha\kappa_\xi+c^2\xi[\alpha]. \tag{33}$$

For constant α,

$$L_{\rm s}\longrightarrow\frac{L_{\rm s}}{\alpha},\qquad P_L\longrightarrow\alpha^2P_L. \tag{34}$$

The area and intrinsic two-geometry do not change. Thus P_L is a well-defined response after a clock prescription has been fixed, but is not an intrinsic scalar of the cut alone. The same is true of energy, temperature, and Hamiltonian charge, which are also defined relative to an evolution vector.

Rindler, Schwarzschild, and spherical dynamics use different clocks but perform the same geometric task. Rindler uses boost flow normalized by the accelerated observer's proper time; Schwarzschild uses Killing time normalized at infinity; spherical dynamics uses the Kodama flow or the Cai–Kim instantaneous thermal calibration. What is common is a prescribed evolution in the two-dimensional normal plane, not a universal observer-independent accelerometer reading.

For a proper-gauge direction X preserving the selected boundary conditions, the source and response descend individually when

$$X[L_{\rm s}] = 0, \qquad X[P_L \epsilon_S] = 0, \tag{35}$$

and the scalar two-form defines a separate reduced sector when it is basic,

$$\iota_X \Omega_L^{(0)} = 0, \qquad \mathcal{L}_X \Omega_L^{(0)} = 0. \tag{36}$$

These conditions hold in the stationary edge-mode sector when clock and cuts are fixed relationally or retained as edge data and only source-preserving degeneracies are quotiented. They fail if arbitrary physical-clock rescalings are declared gauge.

B. Why the standard first law has no additional work term

The usual stationary first law computes the Hamiltonian variation of a specified horizon evolution. On a fixed cut the covariant-phase-space expression has the schematic form

$$\delta H_\xi = \int_S \left(\delta Q_\xi - \xi \cdot \Theta - Q_{\delta\xi}\right), \tag{37}$$

with additional transport terms when the cut moves [17–19]. "Fixed generator" means $\delta\xi^a = 0$ in field space; it does not generally mean $\delta\kappa_\xi = 0$. The complete expression cancels or reallocates pieces caused only by changing the measuring generator, leaving the familiar area channel.

Hayward, Mukohyama, and Ashworth derived the spherical Kodama–Noether charge $\oint Q_K = A\widehat{\kappa}_H/(8\pi)$ in geometric units and explicitly questioned the usual suppression of $\delta\widehat{\kappa}$ [15]. Parattu, Majhi, and Padmanabhan showed that the two pieces of the Einstein surface-Hamiltonian variation become $T\delta S$ and $S\delta T$ in thermodynamically conjugate variables [16]. The acceleration-length representation gives the latter a direct mechanical coordinate:

$$S\,\delta T = -F_L \delta L_{\rm s}. \tag{38}$$

The absence of $S\,dT$ from the standard mass first law is therefore not a missing additive energy. It is the consequence of a different boundary-value problem: the usual law varies the Hamiltonian mass for a specified evolution, whereas Eq. (38) resolves the boundary potential $U_\partial = TS$ when the normal-rate source is varied.

IV. BOUNDARY EQUATION, PULLBACK, AND MECHANICAL–THERMAL SCALE

A. One potential and two pressures

For a stationary uniform cut, integration of the source response gives

$$U_\partial(A, L_{\rm s}) = \mathcal{T}_G \frac{A}{L_{\rm s}}, \qquad \mathcal{T}_G = \frac{c^4}{8\pi G}. \tag{39}$$

Its first derivatives are

$$\Pi_\partial \equiv \left(\frac{\partial U_\partial}{\partial A}\right)_{L_{\rm s}} = \frac{c^4}{8\pi G L_{\rm s}} = \frac{c^2\kappa}{8\pi G}, \tag{40}$$

$$F_L \equiv -\left(\frac{\partial U_\partial}{\partial L_{\rm s}}\right)_A = \frac{c^4 A}{8\pi G L_{\rm s}^2}, \qquad P_L = \frac{F_L}{A}. \tag{41}$$

The quantities are dimensionally and variationally different. Π_∂ is energy per area or force per length, the pressure/tension of a two-dimensional fluid. P_L is force per transverse area, the normal scale pressure.

The pressure interpretation of Π_∂ is independently established in the Damour–Navier–Stokes equation and membrane paradigm, where the tangential momentum balance contains the pressure-gradient term $-D_A\Pi_\partial$ [20–23]. Depending on orientation and stress convention, the same magnitude is called pressure or tension. The pressure interpretation of P_L is instead fixed by Eq. (23). The relation between them is the Hessian identity

$$-\left(\frac{\partial\Pi_\partial}{\partial L_\mathrm{s}}\right)_A = \left(\frac{\partial F_L}{\partial A}\right)_{L_\mathrm{s}} = P_L = \frac{\Pi_\partial}{L_\mathrm{s}}. \tag{42}$$

It is useful to call Π_∂ the *surface pressure/tension* and P_L the *normal scale pressure.*

B. What the pullback statement means

The word pullback has a precise differential-geometric meaning. Fix a nonzero sign branch $\sigma = \mathrm{sgn}(\kappa)$, let S be a horizon cut, and regard the signed acceleration length as a map

$$L_\mathrm{s} : S \longrightarrow \mathbb{R}_\sigma, \qquad x \longmapsto L_\mathrm{s}(x), \qquad \mathbb{R}_\sigma \equiv \{L : \sigma L > 0\}. \tag{43}$$

On that one-dimensional source branch, define

$$\Pi(L) = \frac{c^4}{8\pi G L}, \qquad d\Pi = -P_L(L)\, dL, \qquad P_L(L) = \frac{c^4}{8\pi G L^2}. \tag{44}$$

The pullback $L_\mathrm{s}^*(d\Pi)$ is the one-form on S defined at a point x and tangent vector $X \in T_xS$ by

$$\big[L_\mathrm{s}^*(d\Pi)\big]_x(X) = d\Pi_{L_\mathrm{s}(x)}\big((dL_\mathrm{s})_x(X)\big). \tag{45}$$

In coordinates this is simply the chain rule:

$$\boxed{L_\mathrm{s}^*(d\Pi) = d(\Pi \circ L_\mathrm{s}) = -P_L\, dL_\mathrm{s}, \qquad D_A\Pi_\partial = -P_L D_A L_\mathrm{s}.} \tag{46}$$

On the same fixed-sign branch, $L_\kappa = |L_\mathrm{s}|$ and

$$\Pi_\partial = \sigma P_L L_\kappa, \qquad D_A\Pi_\partial = -\sigma P_L D_A L_\kappa, \qquad |\Pi_\partial| = P_L L_\kappa, \qquad D_A|\Pi_\partial| = -P_L D_A L_\kappa. \tag{47}$$

Consequently the pressure force in the tangential horizon-fluid equation can be written exactly as

$$-D_A\Pi_\partial = P_L D_A L_\mathrm{s} = \sigma P_L D_A L_\kappa. \tag{48}$$

The established membrane pressure-gradient one-form is therefore the pullback to the physical cut of the signed source-space response one-form $-P_L dL_\mathrm{s}$. Equivalently, the stress-magnitude gradient is the pullback of $-P_L dL_\kappa$.

This relation gives P_L an organizing role, but it does not make it ontologically more fundamental than every parent quantity. Π_∂ is the regular surface stress that appears directly in the tangential momentum equation, while P_L is its susceptibility with respect to the mechanically privileged reciprocal-rate coordinate. Moreover, on a generic expanding null boundary the still more general parent source is $\mu = \widehat{\kappa} + \theta/2$. The accurate hierarchy is therefore: μ is the generic null source; Π_∂ is the established linear surface stress on the selected non-expanding sector; and P_L is the normal pressure that generates the change of that stress with acceleration length.

C. A genuine normal force-balance equation

The same relation enters a normal, rather than tangential, projection of Einstein's equation. Freidel and Yokokura showed that a timelike gravitational screen obeys the non-equilibrium equations of a viscous bubble, including a generalized Young–Laplace equation [24]. In their orientation the surface tension is

$$\gamma = -\Pi_\partial = -P_L L_\mathrm{s}. \tag{49}$$

For a static screen the normal momentum balance is

$$0 = \Delta p + \gamma H = \Delta p - \Pi_\partial H, \tag{50}$$

where H is the mean curvature of the cut in the spatial screen and $\Delta p = p_{\rm out} - p_{\rm in}$ includes both matter and the appropriate intrinsic three-curvature contribution. Substitution of Eq. (49) gives the exact signed form

$$\boxed{\Delta p - P_L L_{\rm s} H = 0.} \tag{51}$$

In the positive-κ orientation this is $\Delta p - P_L L_\kappa H = 0$; reversing the screen orientation reverses the signed stress and pressure-jump convention together. Thus P_L enters a genuine normal force balance: its pressure units are converted into curvature pressure by the dimensionless factor $L_{\rm s}H$, or by $L_\kappa H$ in positive magnitudes.

The full dynamical screen equation, in the geometric units and notation of Ref. [24], is

$$-(D_{\boldsymbol{t}} + \theta_{\boldsymbol{t}})\theta_{\boldsymbol{n}} = 8\pi G\big[\Delta p - P_L L_{\rm s} H\big] - \frac{1}{2}\widetilde{\Theta}_{\boldsymbol{t}} : \Theta_{\boldsymbol{n}} - \varrho\,\omega^2 - \Delta\varrho. \tag{52}$$

Here the additional terms encode shear/viscous, normal-connection, and lapse-gradient effects. In a spherically symmetric, irrotational, constant-lapse reduction,

$$-\left(D_{\boldsymbol{t}} + \frac{3}{4}\theta_{\boldsymbol{t}}\right)\frac{\theta_{\boldsymbol{n}}}{8\pi G} = \Delta p - P_L L_{\rm s} H. \tag{53}$$

Equations (51)–(53) are the requested normal force-balance extension. They are exact acceleration-length rewritings of an established gravitational-screen equation, not an additional field equation. They also clarify why no three-dimensional Navier–Stokes equation for P_L is required. The Damour equation governs tangential two-fluid momentum; the Young–Laplace/normal constraint governs normal screen motion. P_L links the two through $\Pi_\partial = P_L L_{\rm s}$, or through $|\Pi_\partial| = P_L L_\kappa$ in positive magnitudes.

D. Mechanical–thermal bridge and entropy density

For the thermodynamic magnitudes in this subsection, choose the positive-κ orientation. Then $L_{\rm s} = L_\kappa$, $\Pi_\partial = |\Pi_\partial|$, and $U_\partial = |U_\partial| = TS$. Reversing orientation leaves T, P_L, L_κ, and all magnitudes unchanged while reversing the signed stress and work form, as summarized in Appendix A. Einstein entropy and the Hawking–Unruh temperature are

$$S = \frac{k_B c^3 A}{4G\hbar} = \frac{k_B A}{4\ell_{\rm P}^2}, \qquad \ell_{\rm P}^2 \equiv \frac{G\hbar}{c^3}, \tag{54}$$

$$T = \frac{\hbar|\kappa|}{2\pi k_B c} = \frac{\hbar c}{2\pi k_B L_\kappa}, \qquad L_\kappa = \frac{c^2}{|\kappa|}. \tag{55}$$

Equation (55) identifies L_κ as a *mechanical–thermal bridge scale*: mechanically it is the reciprocal normal acceleration or boost-rate scale; thermally it is the reduced inverse-temperature length, with Euclidean thermal circumference $2\pi L_\kappa$.

The two laws carried by the same scale are

$$\boxed{P_L(L_\kappa) = \frac{c^4}{8\pi G L_\kappa^2}, \qquad k_B T(L_\kappa) = \frac{\hbar c}{2\pi L_\kappa}.} \tag{56}$$

The first is classical gravitational boundary mechanics and scales as L_κ^{-2}; the second is semiclassical quantum-field temperature and scales as L_κ^{-1}. Their ratio is independent of the clock scale:

$$\boxed{\frac{k_B T}{P_L L_\kappa} = 4\ell_{\rm P}^2.} \tag{57}$$

The boundary potential is separately homogeneous of degree one in A and degree minus one in L_κ:

$$A\left(\frac{\partial U_\partial}{\partial A}\right)_{L_\kappa} = -L_\kappa\left(\frac{\partial U_\partial}{\partial L_\kappa}\right)_A = U_\partial. \tag{58}$$

Its integrated representations form the Euler chain

$$\boxed{U_\partial = TS = \Pi_\partial A = P_L(A L_\kappa) = F_L L_\kappa.} \tag{59}$$

Combining Eqs. (57) and (59) gives

$$\frac{S}{A} = \frac{P_L L_\kappa}{T} = \frac{k_B}{4\ell_{\mathrm{P}}^2}. \tag{60}$$

The factor 1/4 is fixed by the ratio of the Einstein boundary coefficient $1/(8\pi G)$ to the Hawking–Unruh coefficient $1/(2\pi)$; surface gravity and clock normalization cancel. This is not a purely general-relativistic derivation because the temperature contains $\hbar$. It is a complete macroscopic boundary-mechanical reconstruction of the Einstein-horizon entropy density at the interface of classical gravitational action and semiclassical quantum-field temperature.

Ordinary state counting is not a prerequisite for treating this as real entropy. Wald emphasizes that the classical and semiclassical evidence supports black-hole entropy while "it does not seem obvious that black hole entropy corresponds to counting states in any usual sense" [4, 5]. The present equations therefore support a macroscopic informational interpretation: a horizon supplies an observer- or evolution-relative partition between accessible and inaccessible data; boundary mechanics fixes the energy density assigned to that partition; and temperature converts it into entropy. The claim is not that S inventories every detailed fact in the hidden region. Rather, it quantifies information unavailable to, or thermodynamically indistinguishable within, the selected exterior description.

For a black-hole event horizon the partition is causal for the exterior observer. For an eternally accelerated Rindler observer it is causal and permanent for that observer but remains observer-relative because inertial observers do not share it. A generic FRW apparent horizon is a quasi-local marginal screen and need not be a permanent causal barrier; exact de Sitter is the special case in which the apparent and cosmological event horizons coincide.

At fixed area the differential identity is

$$-S\,dT = F_L\,dL_\kappa = P_L\,dV_\perp. \tag{61}$$

Thus gravitational cooling and outward scale work are the same canonical variation in this ensemble. This does not by itself specify an extractable-work protocol.

With

$$\mathcal{V}_\partial \equiv AL_\kappa, \qquad N_S \equiv \frac{S}{k_B}, \tag{62}$$

Eq. (59) becomes

$$\boxed{P_L\mathcal{V}_\partial = N_S k_B T = U_\partial.} \tag{63}$$

This exact ideal-gas-form Euler relation follows from boundary homogeneity, not from assumed molecules.

E. Compliance, modulus, and source-space Hessian

The inverse derivative asked for by dV/dP is a compliance. At fixed transverse area,

$$V_\perp = AL_\kappa, \qquad P_L = \frac{c^4}{8\pi G L_\kappa^2} = \frac{c^4 A^2}{8\pi G V_\perp^2}. \tag{64}$$

Therefore

$$\boxed{\left(\frac{\partial V_\perp}{\partial P_L}\right)_A = -\frac{V_\perp}{2P_L}.} \tag{65}$$

The associated scale compressibility and bulk modulus are

$$\chi_L^{(A)} \equiv -\frac{1}{V_\perp}\left(\frac{\partial V_\perp}{\partial P_L}\right)_A = \frac{1}{2P_L}, \qquad B_L^{(A)} \equiv -V_\perp\left(\frac{\partial P_L}{\partial V_\perp}\right)_A = 2P_L. \tag{66}$$

The positive modulus means that the fixed-area scale direction is mechanically restoring: increasing the normal scale lowers the pressure.

Using Eq. (56),

$$P_L(T) = \frac{\pi c^2 k_B^2}{2G\hbar^2}T^2, \qquad \frac{dP_L}{dT} = \frac{2P_L}{T}, \tag{67}$$

and at fixed area

$$V_\perp T = \frac{A\hbar c}{2\pi k_B}, \qquad \alpha_L^{(A)} \equiv \frac{1}{V_\perp}\left(\frac{\partial V_\perp}{\partial T}\right)_A = -\frac{1}{T}. \tag{68}$$

These are new response functions of the acceleration-length representation.

The Hessian of $U_\partial = \mathcal{T}_G A/L_\kappa$ on the extended source space is

$$\frac{\partial^2 U_\partial}{\partial(A, L_\kappa)^2} = \begin{pmatrix} 0 & -P_L \\ -P_L & 2F_L/L_\kappa \end{pmatrix}, \qquad \det = -P_L^2 < 0. \tag{69}$$

The potential is a saddle if A and L_κ are treated as independent sources. This is not by itself a dynamical instability because the variables define a boundary ensemble, not an unconstrained material state space. The positive fixed-area modulus in Eq. (66) is the relevant local scale response.

V. STATIONARY REALIZATIONS AND THERMAL–MECHANICAL DUALITY

A. Rindler: the operational meaning of acceleration length

For an observer with constant proper acceleration a in Minkowski spacetime, introduce coordinates centered on that observer,

$$cT = \left(\frac{c^2}{a} + \xi\right)\sinh\left(\frac{at}{c}\right), \qquad X = \left(\frac{c^2}{a} + \xi\right)\cosh\left(\frac{at}{c}\right). \tag{70}$$

The observer is at $\xi = 0$ and the Rindler horizon is at $\xi = -c^2/a$. The instantaneous rest slice therefore meets the horizon a proper distance

$$L_\kappa = \frac{c^2}{a} \tag{71}$$

behind the observer. Proper time fixes the physical clock and $\kappa = a$. The source proposition and corner action then give

$$\boxed{P_L^{\mathrm{R}} = \frac{a^2}{8\pi G}, \qquad F_L^{\mathrm{R}} = \frac{Aa^2}{8\pi G}, \qquad U_\partial^{\mathrm{R}} = \frac{Aac^2}{8\pi G}.} \tag{72}$$

No source mass, curvature, center, or spherical volume is used. This is the cleanest local demonstration that the normal pressure is not merely a Newtonian field-stress formula in disguise.

For an observer who accelerates uniformly for all proper time, the Rindler horizon is a permanent causal barrier for that observer. It is nevertheless observer-relative: inertial observers in the same Minkowski spacetime do not share the partition, and a finite-duration accelerated trajectory need not possess the same eternal event horizon. The nonzero response in flat spacetime is therefore attached to the accelerated observer's boundary and clock, not to an observer-independent stress of the Minkowski vacuum.

Frodden, Ghosh, and Perez considered observers held at a small proper distance ℓ outside an isolated horizon and found the local energy

$$E_{\mathrm{FGP}} = \frac{c^4 A}{8\pi G\ell}. \tag{73}$$

To leading order $\ell = c^2/\kappa_{\mathrm{loc}}$, so this is the same local boundary potential, but their construction holds ℓ fixed while varying area [14]. The additional step here is to vary the length as a source and identify its quadratic response.

B. Schwarzschild pressure and field stress

Let $R_H = 2GM/c^2$ and normalize the stationary Killing field to unit time at infinity. Then

$$\kappa_\infty = \frac{GM}{R_H^2} = \frac{c^2}{2R_H} = \frac{c^4}{4GM}, \qquad L_\kappa = 2R_H. \tag{74}$$

With $A_H = 4\pi R_H^2$,

$$\boxed{P_L^{\rm Schw} = \frac{c^4}{32\pi G R_H^2}, \qquad F_L^{\rm Schw} = \frac{c^4}{8G}, \qquad U_\partial^{\rm Schw} = F_L L_\kappa = TS = \frac{Mc^2}{2}.} \tag{75}$$

The force is independent of black-hole size in four spacetime dimensions.

The same pressure has a familiar stationary field-stress correspondence. For a Newtonian gravitational field $\boldsymbol{g} = -\boldsymbol{\nabla}\Phi$, one conventional field-stress tensor is

$$t_{ij}^{(g)} = -\frac{1}{4\pi G}\left(g_i g_j - \frac{1}{2}\delta_{ij}|\boldsymbol{g}|^2\right). \tag{76}$$

For a surface normal parallel to $\boldsymbol{g}$, the longitudinal compressive stress magnitude is

$$P_g^{\parallel} \equiv -t_{nn}^{(g)} = \frac{|\boldsymbol{g}|^2}{8\pi G}. \tag{77}$$

At R_H, the Newtonian field magnitude is $|\boldsymbol{g}_N| = GM/R_H^2 = \kappa_\infty$, hence

$$P_g^{\parallel}(R_H) = P_L^{\rm Schw}. \tag{78}$$

This is a match of stationary, asymptotically normalized magnitudes. It does not turn P_L into a covariant local bulk stress. Rindler already shows that the boundary response is more general than a source-field interpretation [25–27].

C. What it means for the two halves to be distinct

The Schwarzschild Smarr identity can be written

$$\boxed{Mc^2 = TS + F_L L_\kappa, \qquad TS = F_L L_\kappa = \frac{Mc^2}{2}.} \tag{79}$$

This equality holds pointwise for every stationary Schwarzschild state. No time average is required. If a quasi-static history remains on the Schwarzschild family, any time average of the equality follows trivially because the equality already holds at each instant. What has not been derived is the ordinary dynamical virial theorem from the time derivative of a microscopic virial generator. The accurate statement is therefore that Eq. (79) is an exact Euler–Smarr balance with virial form, stronger than a relation that emerges only after averaging.

For comparison, consider an ordinary bounded or periodic Hamiltonian system whose potential is homogeneous of degree n, $V(\lambda q) = \lambda^n V(q)$. The virial theorem gives

$$2\langle K\rangle = n\langle V\rangle, \qquad \langle K\rangle = \frac{n}{n+2}E, \qquad \langle V\rangle = \frac{2}{n+2}E, \tag{80}$$

when the displayed averages and denominator are well defined [28]. The theorem constrains the two averaged components across a family of total energies; it does not require one to decrease whenever the other increases. For $n > 0$, raising E raises both averages. A harmonic oscillator ($n = 2$) has equal halves. A complete ideal projectile flight in uniform gravity, from launch at $z = 0$ to the instant it returns to $z = 0$ before any dissipative impact, has $n = 1$ and $\langle K\rangle = E/3$, $\langle V\rangle = 2E/3$: because the virial function $G = pz$ vanishes at both endpoints, the finite-time boundary term is exactly zero. Repeating the flight with elastic bounces makes the example periodic. The Kepler case $n = -1$ instead has negative binding energy, so adding positive energy makes the orbit less bound and changes the signs and magnitudes differently.

In ordinary mechanics, kinetic and potential energy are called distinct not because they are different substances, but because they are functions of different state-space data and generate different responses: kinetic energy depends on momentum or velocity and controls inertial motion, whereas potential energy depends on configuration and its gradient supplies force. They can be converted into one another while remaining operationally distinguishable energy forms.

The horizon split has an analogous, though not identical, structural basis. On the extended (A, L_κ) source space,

$$TS = A\left(\frac{\partial U_\partial}{\partial A}\right)_{L_\kappa}, \qquad F_L L_\kappa = -L_\kappa\left(\frac{\partial U_\partial}{\partial L_\kappa}\right)_A. \tag{81}$$

The first product is selected by the area/entropy direction and the second by the normal mechanical-scale direction. They are therefore physically distinct conjugate channels even though separate homogeneities force them to equal the same parent potential. On the one-dimensional Schwarzschild solution curve they cannot be adjusted independently, just as a virialized system can have dynamically constrained energy components.

There is an additional exact thermodynamic fact. The formal on-shell Schwarzschild Helmholtz free energy is

$$\boxed{\mathcal{F}_{\mathrm{H}}^{\mathrm{on}} = Mc^2 - TS = \frac{Mc^2}{2} = F_L L_\kappa.} \tag{82}$$

Thus the scale product is precisely the mechanical/free-energy half of the stationary mass. Asymptotically flat Schwarzschild has negative heat capacity and is not a stable unconstrained canonical ensemble, but the on-shell Euclidean free-energy value remains well defined [29].

These results motivate the following interpretation, which is stated as an interpretation rather than a theorem:

> TS is the thermal or kinetic-like half of Schwarzschild energy, while $F_L L_\kappa$ is its mechanical, potential/free-energy-like half. Their exact equality describes a thermal–mechanical virial duality: the redshift-normalized acceleration scale that mechanically supports the horizon is also the scale that thermally excites it.

The qualification matters. In a generic interacting or quantum system TS is not literally the microscopic kinetic energy, and the Rindler realization shows that $F_L L_\kappa$ cannot universally be local curvature or Newtonian potential energy. What is established is the exact split, its two conjugate directions, its free-energy identity, and its co-variation with total mass. No anti-correlated conversion law is required by the virial analogy: ordinary virial components can co-vary when total energy changes, and Schwarzschild does so exactly. The open question is narrower–whether a microscopic horizon description identifies these channels literally as kinetic and potential energies and explains why the macroscopic conjugate split takes this form.

Along the Schwarzschild curve,

$$d(TS) = d(F_L L_\kappa) = dU_\partial = F_L\, dL_\kappa = \frac{1}{2} d(Mc^2), \qquad T\, dS = d(Mc^2). \tag{83}$$

Thus adding black-hole mass increases the thermal and mechanical halves in the same fixed proportion, just as changing the total energy of an ordinary virial family changes its constrained averaged components. The boundary equation coexists exactly with the standard first law: the area leg contributes $2dU_\partial$ and the scale leg subtracts dU_∂, leaving the complete differential dU_∂.

D. Why Schwarzschild and Cai–Kim FRW package energy differently

For any spherical screen of areal radius R carrying

$$E_{\mathrm{sph}} = \frac{c^4 R}{2G}, \tag{84}$$

the ratio between the boundary potential and the spherical charge is

$$\boxed{\frac{U_\partial}{E_{\mathrm{sph}}} = \frac{\kappa R}{c^2}.} \tag{85}$$

Schwarzschild has $\kappa R/c^2 = 1/2$, while the Cai–Kim prescription has $\kappa R/c^2 = 1$. The former therefore contains two copies of the boundary scale U_∂, and the latter one. This is a genuine sectoral difference in how the selected normal rate packages the same spherical charge; it is not merely a choice to rewrite the number two.

A directly measurable thermodynamic consequence appears in the heat capacity along constant-ratio spherical families. If $E = \alpha TS$ and $T \propto R^{-1}$, then

$$C_{\mathrm{phys}} \equiv \frac{dE}{dT} = -\alpha S. \tag{86}$$

Thus Schwarzschild has $C = -2S$, whereas Cai–Kim FRW has $C = -S$. By contrast, in the extended source ensemble at fixed area,

$$C_A^{\mathrm{source}} = \left(\frac{\partial U_\partial}{\partial T}\right)_A = S > 0. \tag{87}$$

The signs differ because the physical spherical families correlate area and temperature, while the source response holds area fixed.

E. Schwarzschild–de Sitter and clock normalization

Schwarzschild hides the normalization issue because asymptotic flatness supplies a conventional clock. For

$$f(r) = 1 - \frac{2GM}{c^2 r} - \frac{\Lambda r^2}{3} \tag{88}$$

and a static generator $\chi^a_\alpha = \alpha(\partial_t)^a$, either horizon r_i has

$$|\kappa_i^{(\alpha)}| = \alpha \frac{c^2}{2r_i} |1 - \Lambda r_i^2|, \qquad P_{L,i}^{(\alpha)} = \frac{[\kappa_i^{(\alpha)}]^2}{8\pi G}. \tag{89}$$

Different admissible normalizations assign different numerical responses to the same intrinsic cut. Once a particular static observer is selected, the locally redshifted value is consistent. The absence of a preferred global clock is therefore a physical ensemble issue, not an algebraic inconsistency [38, 39].

VI. FRW SCREEN WORK AND NORMAL FOCUSING

A. Compact-screen response

Consider the four-dimensional FRW metric

$$ds^2 = -c^2 dt^2 + a^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega_2^2 \right), \qquad R(t, r) = a(t) r, \tag{90}$$

and let ρ and $p_{\rm m}$ denote total energy density and matter pressure in the same units. The apparent-horizon radius is

$$R_A = \frac{c}{\sqrt{H^2 + kc^2/a^2}}. \tag{91}$$

The Friedmann constraint and Misner–Sharp definition give

$$E_A = \rho V_A = \frac{c^4 R_A}{2G}, \qquad V_A = \frac{4\pi}{3} R_A^3, \qquad \rho = \frac{3c^4}{8\pi G R_A^2}. \tag{92}$$

Along the homogeneous family of apparent-horizon screens,

$$\boxed{P_{\rm scr} \equiv \frac{dE_A}{dV_A} = \frac{c^4}{8\pi G R_A^2} = \frac{\rho}{3}.} \tag{93}$$

This is a positive compact-screen size response, not the cosmic-fluid pressure $p_{\rm m}$ and not the conventional independent-variable derivative $-(\partial E/\partial V)_S$. Both S_A and V_A are functions of the single physical parameter R_A.

The associated force is

$$F_{\rm scr} = A_A P_{\rm scr} = \frac{c^4}{2G}, \qquad dE_A = F_{\rm scr}\, dR_A = P_{\rm scr}\, dV_A. \tag{94}$$

The force is again constant in four dimensions, now four times the Schwarzschild value because the Cai–Kim acceleration scale is twice the Schwarzschild surface gravity at the same radius.

The inverse pressure slope along the physical spherical family is

$$\boxed{\frac{dV_A}{dP_{\rm scr}} = -\frac{3V_A}{2P_{\rm scr}}, \qquad B_{\rm scr} \equiv -V_A \frac{dP_{\rm scr}}{dV_A} = \frac{2}{3} P_{\rm scr}.} \tag{95}$$

The factor of three relative to Eq. (65) arises because the transverse area changes with the radius rather than being held fixed.

B. Cai–Kim equilibrium scale

The instantaneous Cai–Kim prescription is

$$\kappa_{\rm CK} = \frac{c^2}{R_A}, \qquad L_{\rm CK} = R_A, \qquad T_{\rm CK} = \frac{\hbar c}{2\pi k_B R_A}. \tag{96}$$

It makes the acceleration-length pressure coincide exactly with the screen response:

$$P_L^{\rm CK} = P_{\rm scr}, \qquad T_{\rm CK} S_A = E_A. \tag{97}$$

The differential chain is

$$T_{\rm CK} dS_A = 2dE_A, \qquad -S_A dT_{\rm CK} = F_L^{\rm CK} dL_{\rm CK} = P_{\rm scr} dV_A = dE_A, \tag{98}$$

and hence

$$\boxed{dE_A = T_{\rm CK} dS_A - P_{\rm scr} dV_A.} \tag{99}$$

This is the cleanest literal $P\,dV$ realization of the normal scale response. Every quantity is fixed by the same macroscopic radius. The result is an exact reorganization of the Einstein–FRW constraint, not a new Friedmann equation and not Cai and Kim's original matter-flux identity [30].

The distinction matters. Cai and Kim used the energy supply crossing a momentarily fixed apparent horizon,

$$\delta Q_{\rm matt} = A_A(\rho + p_{\rm m}) H R_A\, dt = T_{\rm CK} dS_A, \tag{100}$$

with signs determined by orientation. Equation (99) instead resolves the simultaneous change of the constrained screen state into entropy and size legs. The flux law sorts change by carrier; the screen law sorts the same geometry by boundary coordinate.

The sectoral heat capacity from Eq. (86) is

$$C_{\rm CK} = \frac{dE_A}{dT_{\rm CK}} = -S_A, \tag{101}$$

whereas Schwarzschild has $C = -2S$. This is one concrete way in which the $U_\partial = E$ versus $U_\partial = E/2$ distinction manifests rather than merely renames the energy.

C. Full Hayward surface gravity

The Cai–Kim scale is an instantaneous equilibrium calibration. The signed Kodama–Hayward surface gravity probes the evolving normal two-geometry:

$$\kappa_H^{\rm s} = \frac{c^2}{2} \Box_h R\big|_{R_A} = -\frac{c^2}{R_A}\left(1 - \frac{\dot{R}_A}{2HR_A}\right). \tag{102}$$

It measures the transverse-null slope of the marginal condition, or normal cross-focusing. A comoving FRW observer is geodesic, so this is not the observer's proper acceleration.

Using the Friedmann and continuity equations,

$$\frac{\dot{R}_A}{2HR_A} = \frac{3}{4}(1+w), \qquad w \equiv \frac{p_{\rm m}}{\rho}, \tag{103}$$

for arbitrary spatial curvature. Therefore

$$\kappa_H^{\rm s} = \frac{c^2}{4R_A}(3w - 1), \tag{104}$$

and its quadratic response is

$$\boxed{P_L^H = \frac{(\kappa_H^{\rm s})^2}{8\pi G} = \left(\frac{1-3w}{4}\right)^2 P_{\rm scr}.} \tag{105}$$

Thus P_L^H and $P_{\rm scr}$ answer different questions. For dust, $P_L^H = P_{\rm scr}/16$; for radiation it vanishes; and for de Sitter it equals $P_{\rm scr}$.

Faraoni showed that the expanding apparent horizon changes trapping and oriented-temperature character at

$$p_{\rm m} = \frac{\rho}{3} = P_{\rm scr}. \tag{106}$$

The threshold is established work; the additional reading here is that it equates matter pressure with the independently defined compact-screen response. At the threshold, $\kappa_H^{\rm s} = 0$ and $P_L^H = 0$ while $P_{\rm scr}$ remains finite. The rate variable remains regular, but its reciprocal length chart reaches infinity [33].

A generic FRW apparent horizon is a quasi-local marginal screen rather than a cosmological event horizon. Its worldtube can be timelike, spacelike, or null, and it can be crossed as the cosmology evolves. Exact de Sitter is the special case in which the apparent and event horizons coincide.

D. Hayward–Mukohyama–Ashworth parent potential

Hayward's work density is

$$W = -\frac{1}{2}T^{ab}h_{ab} = \frac{\rho - p_{\rm m}}{2}. \tag{107}$$

The spherical Einstein equations and the Hayward–Mukohyama–Ashworth Kodama–Noether charge give

$$\widetilde{U_\partial} \equiv \frac{A_A c^2 \kappa_H^{\rm s}}{8\pi G} = \frac{E_A}{2} - \frac{3}{2}WV_A. \tag{108}$$

Pulling a variation back along the physical trapping-horizon trajectory, Hayward's unified first law is

$$d_H E_A = T_H^{\rm s} d_H S_A + W\, d_H V_A. \tag{109}$$

For nonzero $\kappa_H^{\rm s}$, differentiation of Eq. (108) gives the exact allocation

$$\boxed{-F_L^H d_H L_H^{\rm s} = -\frac{1}{2}T_H^{\rm s} d_H S_A - W\, d_H V_A - \frac{3}{2}V_A d_H W.} \tag{110}$$

This shows precisely how the reciprocal-rate variation is redistributed among area, matter-work, and changing-work-density terms. It is the appropriate dynamical qualification to the equilibrium Cai–Kim law [15, 31].

VII. INTEGRABILITY AND FINITE DYNAMICAL BALANCE

A. When the scale leg is and is not a state energy

On the extended boundary source space, define the scale one-form

$$\alpha_L \equiv -F_L\, \boldsymbol{\delta} L_{\rm s} = \frac{Ac^2}{8\pi G}\, \boldsymbol{\delta}\kappa. \tag{111}$$

Its phase-space exterior derivative is

$$\boxed{\mathbf{d}_\Gamma \alpha_L = \frac{c^2}{8\pi G}\boldsymbol{\delta} A \wedge \boldsymbol{\delta}\kappa = -P_L \boldsymbol{\delta} A \wedge \boldsymbol{\delta} L_{\rm s}.} \tag{112}$$

Therefore there is no universal state function $H_L(A, L_{\rm s})$ satisfying $\boldsymbol{\delta} H_L = \alpha_L$ when area and scale vary independently. This does not disqualify the term as work; ordinary $P\,dV$ is generally nonexact. It rules out only a separately stored scale energy on the unrestricted two-dimensional source space.

The complete parent variation remains exact because

$$\mathbf{d}_\Gamma(\Pi_\partial \boldsymbol{\delta} A) = -\mathbf{d}_\Gamma \alpha_L, \qquad \boldsymbol{\delta} U_\partial = \Pi_\partial \boldsymbol{\delta} A + \alpha_L. \tag{113}$$

On a one-dimensional physical family, or any leaf $L_{\rm s} = L_{\rm s}(A)$, the pullback of the two-form vanishes and the scale leg is integrable along that leaf. Ashtekar, Fairhurst, and Krishnan obtained the equivalent condition $\boldsymbol{\delta}\kappa \wedge \boldsymbol{\delta}A = 0$ for isolated-horizon Hamiltonian evolution [34].

For spherical Hayward horizons, using

$$\kappa_H^{\rm s} = \frac{c^2}{2R} - \frac{4\pi GR}{c^2}W \tag{114}$$

in Eq. (112) gives the exact reduced obstruction

$$\boxed{\mathbf{d}_\Gamma \alpha_L = -\boldsymbol{\delta}V \wedge \boldsymbol{\delta}W.} \tag{115}$$

The isolated scale leg is therefore integrable on restricted leaves such as fixed W, $W = W(V)$, or a single history, but not for generic independent variations of screen size and equation of state.

B. Finite constant-force law for dynamical black holes

Ashtekar, Paraizo, and Shu proved an exact nonperturbative charge–flux law for quasi-local dynamical black-hole horizons that can be arbitrarily far from equilibrium [35]. Their preferred equilibrium projection assigns each marginal cut the Kerr surface gravity $\bar\kappa(A, J)$. In their geometric units the phase-space charge is $Q_{\rm PS} = \bar\kappa A/(8\pi G)$ and obeys

$$\Delta Q_{\rm PS} = \mathfrak{F}^{(\xi)}_{\rm gws} + \mathfrak{F}^{(\xi)}_{\rm matt}. \tag{116}$$

Restoring acceleration units and defining

$$L_{\rm APS} = \frac{c^2}{\bar\kappa}, \qquad U_{\rm APS} = \frac{Ac^2\bar\kappa}{8\pi G}, \qquad F_L^{\rm APS} = \frac{A\bar\kappa^2}{8\pi G}, \tag{117}$$

gives the exact acceleration-length representation

$$\boxed{\Delta U_{\rm APS} = \int_{S_1}^{S_2} \left(\Pi_\partial \, dA - F_L^{\rm APS} dL_{\rm APS}\right) = E^{(\xi)}_{\rm gws} + E^{(\xi)}_{\rm matt}.} \tag{118}$$

APS supply the second equality; the first is the present source representation. Matter and gravitational fluxes do not separately map onto the area and scale legs.

In the nonrotating sector the equilibrium projection is Schwarzschild:

$$\bar\kappa = \frac{c^2}{2R}, \qquad L_{\rm APS} = 2R, \qquad U_{\rm APS} = \frac{c^4 R}{4G}, \qquad F_L^{\rm APS} = \frac{c^4}{8G}. \tag{119}$$

The force is exactly constant. Consequently the finite physical-process law becomes

$$\boxed{E^{(\xi)}_{\rm gws} + E^{(\xi)}_{\rm matt} = \Delta U_{\rm APS} = F_L^{\rm APS}\, \Delta L_{\rm APS}.} \tag{120}$$

This is a genuine dynamical force–displacement law, not merely a stationary differential identity.

APS normalize the nonrotating horizon mass as $E_{\rm DH} = 2U_{\rm APS}$. Since $U_{\rm APS} = TS = F_L^{\rm APS} L_{\rm APS}$ on every projected cut,

$$\boxed{\Delta(TS) = \Delta(F_L L) = \Delta U_{\rm APS} = \frac{1}{2}\Delta E_{\rm DH}.} \tag{121}$$

Thus the thermal and mechanical halves co-evolve equally during a finite nonrotating dynamical-horizon process. This is the strongest current dynamical support for the Schwarzschild duality and is exactly the behavior expected when added total energy changes two constrained virial components in a fixed ratio. It does not identify one half with matter flux and the other with gravitational-wave flux. What remains interpretive is their literal microscopic kinetic/potential meaning, not their equal co-evolution, which is exact.

For rotating cuts, A, J, $\bar\kappa$, and $\bar\Omega$ are linked by the Kerr projection and the rotational channel must be retained. The constant-force simplification is specific to the nonrotating four-dimensional sector.

VIII. STATIONARY WALD-ENTROPY AND HIGHER-CURVATURE EXTENSION

The acceleration-length idea has a clean stationary extension, but two notions must be separated: the universal $S_W\,\delta T$ channel and the specific membrane pressure of a higher-curvature theory.

Chakraborty and Padmanabhan showed that the null-surface variation of the Lanczos–Lovelock surface term can again be organized as $S_W\delta T$ and $T\delta S_W$, with S_W the appropriate Wald entropy [36]. Holding Wald entropy fixed and using $T = \hbar c/(2\pi k_B L_\kappa)$ gives

$$S_W\,dT = -F_L^{(W)} dL_\kappa, \qquad \boxed{F_L^{(W)} = \frac{TS_W}{L_\kappa}, \qquad P_{L,SdT}^{(W)} = \frac{F_L^{(W)}}{A} = \frac{\hbar c}{2\pi k_B L_\kappa^2} s_W,} \tag{122}$$

where $s_W = S_W/A$. For Einstein gravity, $s_W = k_B/(4\ell_{\rm P}^2)$ and Eq. (122) reduces exactly to $\kappa^2/(8\pi G)$.

The membrane pressure in a pure mth-order Lanczos–Lovelock theory has an additional theory- and dimension-dependent factor. Kolekar and Kothawala found

$$\frac{\Pi_\partial^{(m)} A}{T} = \frac{D-2m}{D-2} S_W^{(m)}. \tag{123}$$

At fixed Wald entropy density and couplings, its acceleration-length susceptibility is therefore

$$\boxed{P_{L,\rm mem}^{(m)} = -\left(\frac{\partial \Pi_\partial^{(m)}}{\partial L_\kappa}\right)_{s_W} = \frac{D-2m}{D-2}\frac{\hbar c}{2\pi k_B L_\kappa^2} s_W^{(m)}.} \tag{124}$$

For a sum of Lovelock densities, the corresponding stationary membrane response is the sum of the weighted entropy densities. Einstein gravity is the $m = 1$ case and the factor equals one. In the critical dimension $D = 2m$, the pure Lovelock term is topological: it can contribute a global Wald entropy while its local membrane pressure factor vanishes. This is a useful check rather than a paradox [37].

The distinction between Eqs. (122) and (124) is important. The first is the response of the thermodynamic $S_W\delta T$ channel. The second is the derivative of the actual Lanczos–Lovelock membrane pressure. They coincide in Einstein gravity but not universally in higher-curvature theories. Moreover, fixed area need not mean fixed Wald entropy. Along a physical family for which S_W changes with L_κ,

$$-\frac{d(TS_W)}{dL_\kappa} = \frac{TS_W}{L_\kappa} - T\frac{dS_W}{dL_\kappa}, \tag{125}$$

so the second term belongs to the entropy/area sector rather than the fixed-entropy scale response.

For stationary $f(R)$ gravity with f_R effectively constant over the source variation, $s_W = f_R k_B/(4\ell_{\rm P}^2)$ and the Wald-channel result becomes

$$P_{L,SdT}^{f(R)} = f_R\frac{\kappa^2}{8\pi G}. \tag{126}$$

Generic variations of f_R and generic dynamical higher-curvature horizons add derivative and non-Newtonian membrane terms. A complete covariant null-source proof in those theories remains separate work. The result established here is the stationary Wald-channel lift and its precise relation to the known Lovelock membrane equation of state.

IX. GENERALITY, ANTECEDENTS, AND INTRINSIC SIGNIFICANCE

A. What is universal and what is prescription dependent

The construction has three levels:

1. The generic covariant Einstein null scalar source is $\mu = \widehat{\kappa} + \theta/2$.

2. On a clock-fixed non-expanding null sector, $\mu = \widehat{\kappa}$ and the branchwise coordinate change to $L_{\rm s} = 1/\widehat{\kappa}$ gives the exact Hamilton–Jacobi response $P_L = \kappa^2/(8\pi G)$.

3. In other sectors, an independently established normal-rate prescription and parent $A\kappa$ potential can support the same algebra, but not automatically the same reduced canonical theorem.

Squaring removes the orientation sign, not the clock dependence. Extremal or trace-boundary points with $\kappa = 0$ must be described in the regular parent rate rather than the reciprocal chart.

The pullback and normal-balance results sharpen the physical hierarchy. The generic source μ is the broadest null variable. The linear surface pressure/tension Π_∂ appears directly in tangential and normal screen equations. The quadratic P_L is the normal susceptibility of that stress with respect to the mechanically privileged acceleration length. It is not more general than μ, but it is more directly analogous to ordinary three-dimensional pressure because it is force per area and multiplies swept normal volume.

In D spacetime dimensions, if a spherical sector has $A \propto R^{D-2}$ and $L_\kappa \propto R$, then

$$F_L = AP_L \propto R^{D-4}. \tag{127}$$

The scale-independent force is therefore special to four spacetime dimensions. This dimensional fact is independent of the numerical normalization selected in a particular sector.

B. Relation to prior work

The ingredients on which the construction rests are established: area–boost conjugacy; local near-horizon A/ℓ energy; the $A\kappa$ Kodama–Noether charge; the $T\delta S + S\delta T$ surface-term split; the null scalar pair and mixed boundary conditions; the Damour–Navier–Stokes membrane pressure; Freidel and Yokokura's Young–Laplace screen equation; Hayward's unified first law; the Cai–Kim apparent-horizon construction; and the APS finite dynamical charge–flux theorem [7–9, 12–16, 20, 21, 24, 30, 31, 35].

The novelty is not the existence of the $A\kappa$ potential, the $S\,dT$ channel, the linear membrane pressure, the Bekenstein–Hawking factor four, the Young–Laplace equation, or the APS charge. The added construction is the acceleration-length source representation and the structure it exposes:

1. the action and corner derivations of $P_L = \kappa^2/(8\pi G)$ and $F_L = AP_L$;
2. the direct pressure-work form $-P_L \delta V_\perp$;
3. the signed pullback relation $D_A \Pi_\partial = -P_L D_A L_\mathrm{s}$ (equivalently $D_A|\Pi_\partial| = -P_L D_A L_\kappa$) and the normal balance $\Delta p - P_L L_\mathrm{s} H = 0$;
4. the mechanical–thermal bridge laws $P_L \propto L_\kappa^{-2}$ and $T \propto L_\kappa^{-1}$;
5. the exact Schwarzschild thermal–mechanical/free-energy duality;
6. the positive Cai–Kim FRW screen pressure and compact first law;
7. the response functions and integrability criterion; and
8. the nonrotating APS finite constant-force flux law and the stationary Wald-entropy lift.

The contribution belongs to the same scientific archetype as the membrane reformulation of horizon dynamics, Iyer–Wald's Noether-charge organization, Jacobson's equation-of-state derivation, Hayward's unified first law, Padmanabhan's horizon thermodynamic identity, Cai–Kim's FRW construction, and the Frodden–Ghosh–Perez local first law: established gravity is reorganized so that a previously obscured thermodynamic or mechanical structure becomes explicit [6, 14, 17, 30–32]. This is a comparison of contribution type, not a claim that the present work has already attained the historical reach of those papers.

To our knowledge, prior work has not combined the mixed null rate-source polarization, the branchwise coordinate $L_\kappa = c^2/|\kappa|$, the quadratic response $\kappa^2/(8\pi G)$, its direct normal-volume work, and the Rindler–Schwarzschild–FRW–APS dictionary developed here. Equivalent variables can be hidden by conventions, so the priority claim remains appropriately qualified.

The positive FRW screen-work result was publicly disclosed by the author on May 10, 2025: `https://x.com/JTT329/status/1921341959009902709`.

X. CONCLUSION

Einstein gravity's regular null scalar source is generically $\mu = \widehat{\kappa} + \theta/2$. On a clock-anchored non-expanding sector it reduces to surface gravity, and reciprocal acceleration length is an admissible branchwise source coordinate. The direct symplectic-potential and independent area–boost results are

$$\Theta_L^{(0)} = -\int d\tau \int_S \epsilon_S \, \frac{\kappa^2}{8\pi G}\delta L_{\rm s}, \qquad P_L = \frac{\kappa^2}{8\pi G}, \qquad F_L = AP_L. \tag{128}$$

At fixed transverse area, the term is $-P_L\delta V_\perp$, so the response is pressure in the ordinary mechanical sense.

The linear membrane pressure and the normal pressure are different but linked. In signed source variables,

$$\Pi_\partial = P_L L_{\rm s}, \qquad D_A\Pi_\partial = -P_L D_A L_{\rm s}; \tag{129}$$

while in positive magnitudes $|\Pi_\partial| = P_L L_\kappa$ and $D_A|\Pi_\partial| = -P_L D_A L_\kappa$. The differential relation is the precise pullback of the source-space response to a nonuniform horizon cut. The same identity rewrites the Young–Laplace screen constraint as $\Delta p - P_L L_{\rm s} H = 0$, or $\Delta p - P_L L_\kappa H = 0$ in the positive-κ orientation, supplying a genuine normal force balance.

Acceleration length is also a mechanical–thermal bridge scale:

$$\boxed{L_\kappa = \frac{c^2}{|\kappa|} = \frac{\hbar c}{2\pi k_B T}, \qquad P_L(L_\kappa) = \frac{c^4}{8\pi G L_\kappa^2}.} \tag{130}$$

The ratio of the classical pressure law to the semiclassical temperature law reconstructs one entropy nat–one unit of S/k_B–per $4\ell_{\rm P}^2$. With Wald's caution that black-hole entropy need not correspond to ordinary state counting, the result supports a macroscopic informational interpretation: a horizon selects an accessible/inaccessible partition, gravitational boundary mechanics assigns it an energy density, and temperature converts that density into entropy.

Schwarzschild realizes an exact pointwise thermal–mechanical virial duality,

$$Mc^2 = TS + F_L L_\kappa, \qquad TS = F_L L_\kappa = \mathcal{F}_{\rm H}^{\rm on} = \frac{Mc^2}{2}. \tag{131}$$

The two products are distinct conjugate channels but not independently adjustable reservoirs on the one-parameter solution family. Their variations satisfy $d(TS) = d(F_L L_\kappa) = \frac{1}{2}d(Mc^2)$, so added mass raises both halves in lockstep, just as the constrained components of an ordinary virial family can co-vary when its total energy changes. Interpreting TS as kinetic/thermal-like and $F_L L_\kappa$ as potential/free-energy-like is physically motivated and stated explicitly. What remains conjectural is the literal microscopic identification, not the macroscopic co-evolution.

Cai–Kim FRW gives the most literal spherical application: $P_L = P_{\rm scr} = \rho/3$ and $dE_A = T_{\rm CK}dS_A - P_{\rm scr}dV_A$. Full Hayward surface gravity instead measures cross-focusing and obeys the HMA redistribution law. Beyond equilibrium, the scale leg is generally nonintegrable when area and scale vary independently, but the complete parent charge remains exact. For nonrotating APS dynamical horizons, the same structure becomes the finite constant-force law

$$E_{\rm gws}^{(\xi)} + E_{\rm matt}^{(\xi)} = \frac{c^4}{8G}\,\Delta L_{\rm APS}. \tag{132}$$

A stationary higher-curvature extension replaces the Einstein entropy density by the appropriate Wald density, with a separate Lovelock-dependent membrane factor.

The central result is therefore simple but not merely algebraic: Einstein gravity supplies a normal-rate boundary source; acceleration length is its natural mechanical–thermal coordinate; and surface gravity squared is the action-selected normal pressure in that coordinate. The representation exposes a single structure as local Rindler mechanics, Schwarzschild thermal–mechanical duality, FRW screen work, and finite dynamical force–displacement balance, while stating exactly where clock choice, higher-curvature dynamics, and integrability limit the claim.

Appendix A: Signed and positive acceleration length

The canonical derivation uses the signed source $L_{\rm s} = c^2/\kappa$. On a fixed-sign branch define

$$\sigma \equiv {\rm sgn}(\kappa), \qquad L_\kappa = \frac{c^2}{|\kappa|} = |L_{\rm s}|, \qquad L_{\rm s} = \sigma L_\kappa. \tag{A1}$$

The signed boundary potential and surface stress are then

$$U_\partial = \sigma \frac{Ac^4}{8\pi G L_\kappa}, \qquad \Pi_\partial = P_L L_\mathrm{s} = \sigma P_L L_\kappa, \qquad |\Pi_\partial| = P_L L_\kappa. \tag{A2}$$

Their spatial derivatives obey

$$D_A \Pi_\partial = -P_L D_A L_\mathrm{s} = -\sigma P_L D_A L_\kappa, \qquad D_A|\Pi_\partial| = -P_L D_A L_\kappa, \tag{A3}$$

and the source work form becomes

$$\Theta_L^{(0)} = -\sigma \int d\tau \int_S \epsilon_S \, P_L \delta L_\kappa. \tag{A4}$$

Thus P_L and all positive magnitudes are orientation independent, while the signed stress and work one-form carry σ. Neither reciprocal coordinate covers $\kappa = 0$.

Appendix B: FRW rate identities

Einstein's equations in the conventions of Eq. (90) are

$$H^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3c^2}\rho, \tag{B1}$$

$$\dot{H} - \frac{kc^2}{a^2} = -\frac{4\pi G}{c^2}(\rho + p_\mathrm{m}), \tag{B2}$$

$$\dot{\rho} + 3H(\rho + p_\mathrm{m}) = 0. \tag{B3}$$

Differentiating Eq. (91) gives

$$\dot{R}_A = \frac{4\pi G}{c^4} H R_A^3 (\rho + p_\mathrm{m}), \tag{B4}$$

and hence

$$\frac{\dot{R}_A}{2HR_A} = \frac{2\pi G R_A^2}{c^4}(\rho + p_\mathrm{m}) = \frac{3}{4}(1+w), \tag{B5}$$

proving Eq. (103) for arbitrary k.

The spherical surface-gravity identity

$$\kappa_H^\mathrm{s} = \frac{GE_A}{c^2 R_A^2} - \frac{4\pi G R_A}{c^2} W \tag{B6}$$

immediately gives Eq. (108); differentiating it and using Eq. (109) gives Eq. (110).

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