Horizon Pressure from the Einstein Boundary Potential

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Abstract

Within the standard prescribed-clock setting of horizon mechanics, the Einstein boundary potential contains a transverse area direction and a normal surface-gravity direction. Expressing the latter through the signed acceleration length $L_s = c^2/\kappa$ makes the full differential separate into the familiar surface-stress area term and a normal force-displacement term. The resulting force per transverse area is $P_L = \kappa^2/(8\pi G)$ and its displacement gives pressure–volume work. This coefficient appears directly in the Einstein null-boundary symplectic potential as the canonical response to an acceleration-length source variation, and the stationary area–boost corner pair yields the same force independently. Rindler fixes the operational normalization of the length; Schwarzschild gives an exact thermal–mechanical Smarr split; and the instantaneous-equilibrium FRW apparent horizon gives $P_{\mathrm{scr}} = \rho/3$ and a macroscopic screen-state first law. Hawking–Unruh thermality is introduced only afterward and identifies the same positive scale as the reduced thermal length. The result establishes a normal mechanical sector of the Einstein horizon boundary structure without modifying the field equations or introducing a material stress tensor.

Keywords: horizon thermodynamics, surface gravity, acceleration length, gravitational boundary pressure, black holes, FRW cosmology

1. Introduction and motivation

Horizon mechanics singles out area and surface gravity. The Bardeen–Carter–Hawking first law contains the area coefficient $c^2\kappa/(8\pi G)$, Smarr's integrated relation contains the corresponding $A\kappa$ structure, and later membrane and boundary-action formulations supplied surface-stress and energy-valued boundary interpretations of the same data [1, 2, 3, 4, 5]. The scalar boundary potential used here is

$$\boxed{U_\partial(A,\kappa) = \frac{Ac^2\kappa}{8\pi G}}. \tag{1}$$

The word *energy-valued* is intentional: U_∂ has the dimensions and boundary-Hamiltonian role of an energy, but it is not asserted to be the total energy of a spacetime. In Schwarzschild $U_\partial = Mc^2/2$; in the Cai–Kim FRW realization U_∂ equals the Misner–Sharp horizon energy; and in Rindler it is a finite-patch boundary potential without an enclosed source mass.

The potential contains two geometric directions. The transverse direction is the cut area A, whose coefficient is the familiar two-dimensional horizon surface pressure or tension. The normal direction is the rate κ of the selected horizon flow. This raises a mechanical question: if the transverse direction is paired with a surface stress, is there a natural displacement coordinate for the normal-rate direction, together with a conjugate force and pressure?

The question is especially sharp in cosmology. In the conclusion to his discussion of FLRW apparent-horizon thermodynamics, Faraoni noted that "the work term appearing in the first law is somehow introduced *a posteriori* and, if one did not already know the first law, it is unlikely that one would discover it this way" [6]. The construction below produces the work term mechanically before a thermodynamic first law is imposed.

Throughout, the setting is the standard one in which the physical horizon flow and clock have been prescribed and an equilibrium or nonexpanding sector appropriate to the realization has been selected. This is the same kind of normalization required to define horizon energy, temperature, and surface gravity in the first place.

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2. Acceleration length and the mechanical normal form

On either nonzero signed branch define

$$\Omega_\kappa \equiv \frac{|\kappa|}{c}, \qquad L_s \equiv \frac{c^2}{\kappa}, \qquad L_\kappa \equiv |L_s| = \frac{c}{\Omega_\kappa} = \frac{c^2}{|\kappa|}. \quad (2)$$

The definition has a physical but sector-dependent realization. For a stationary normal flow, Ω_κ is the inverse-time scale associated with the normalized boost or inaffinity rate, and $L_\kappa = c/\Omega_\kappa$ is its reciprocal light-travel scale. Rindler geometry fixes the normalization operationally: a uniformly accelerated observer with $\kappa = a$ lies an exact proper distance c^2/a from the Rindler horizon [7]. In Schwarzschild, Killing time normalized at infinity gives $L_\kappa = 2R_H$. In the instantaneous Cai–Kim FRW prescription, $\kappa_{\rm CK} = c^2/R_A$ gives $L_\kappa = R_A$.

Up to a dimensionless constant, c^2/κ is the only length constructible from c and κ without another scale; Rindler fixes that constant to unity. The Schwarzschild and FRW relations are realizations of the same reciprocal normal-rate scale rather than definitions imposed from spherical symmetry.

Equation (1) becomes

$$\boxed{U_\partial(A, L_s) = \frac{Ac^4}{8\pi G L_s}}. \quad (3)$$

Its full differential is

$$\begin{aligned} dU_\partial &= \frac{c^4}{8\pi G L_s}\, dA - \frac{Ac^4}{8\pi G L_s^2}\, dL_s \\ &= \frac{U_\partial}{A}\, dA - \frac{U_\partial}{L_s}\, dL_s. \end{aligned} \quad (4)$$

Define

$$\Pi_\partial \equiv \left(\frac{\partial U_\partial}{\partial A}\right)_{L_s} = \frac{U_\partial}{A} = \frac{c^2\kappa}{8\pi G}, \quad (5)$$

$$F_L \equiv -\left(\frac{\partial U_\partial}{\partial L_s}\right)_A = \frac{U_\partial}{L_s} = \frac{A\kappa^2}{8\pi G}, \quad (6)$$

$$P_L \equiv \frac{F_L}{A} = \frac{\kappa^2}{8\pi G} = \frac{c^4}{8\pi G L_s^2}. \quad (7)$$

With $M_\partial \equiv U_\partial/c^2$, the force takes the familiar form

$$\boxed{F_L = M_\partial \kappa.} \quad (8)$$

At fixed transverse area, introduce the swept normal-volume displacement

$$dV_\perp \equiv A\, dL_s. \quad (9)$$

Then

$$\boxed{dU_\partial = \Pi_\partial\, dA - P_L\, dV_\perp, \qquad F_L\, dL_s = P_L\, dV_\perp.} \quad (10)$$

This is the direct mechanical reason for the word pressure: P_L is force per transverse area and multiplies a normal volume displacement. The known surface stress is resolved as

$$\boxed{\Pi_\partial = P_L L_s, \qquad |\Pi_\partial| = P_L L_\kappa, \qquad P_L = -\frac{\partial|\Pi_\partial|}{\partial L_\kappa}.} \quad (11)$$

The two source directions also carry equal Euler weight,

$$A\left(\frac{\partial U_\partial}{\partial A}\right)_{L_s} = -L_s\left(\frac{\partial U_\partial}{\partial L_s}\right)_A = U_\partial. \quad (12)$$

In this precise variational sense, (A, Π_∂) and (L_s, F_L) are complementary conjugate pairs of the same energy-valued potential. Area is intrinsic to the horizon cut; acceleration length is the normal scale defined after the physical flow and clock are fixed.

It is useful to display the Einstein coupling

$$\kappa_{\rm E} \equiv \frac{8\pi G}{c^4}. \quad (13)$$

Equations (5) and (7) then imply

$$\boxed{\kappa_{\rm E}\Pi_\partial L_s = 1, \qquad \kappa_{\rm E} P_L L_s^2 = 1.} \quad (14)$$

The same coupling that normalizes Einstein's field equation therefore converts one inverse power of the normal length into surface stress and two inverse powers into normal pressure.

3. Canonical origin of the response

The preceding derivation already identifies a conjugate force from an energy-valued potential. A stronger question is whether the normal-rate variation is genuine gravitational boundary source data, or whether the force has only been manufactured by a convenient change of variables. The null-boundary symplectic structure answers that question independently.

Let $\mathcal{N}$ be a null hypersurface foliated by spacelike cuts S, and let ξ^a be its selected null evolution vector. Hopfmüller and Freidel showed that the scalar sector of the Einstein null-boundary symplectic structure contains the mixed source [8, 9]

$$\mu_\xi = \widehat{\kappa}_\xi + \frac{1}{2}\theta_\xi, \quad (15)$$

where $\widehat{\kappa_\xi}$ is the inverse-length inaffinity and θ_ξ the null expansion. In the rate-source polarization, the scalar boundary one-form may be written schematically as

$$\Theta_\mu^{(0)} = \frac{1}{8\pi G} \int_N \epsilon_S \, \delta\mu_\xi \, dv, \tag{16}$$

with ϵ_S the area element of a cut. Later covariant treatments likewise identify the corresponding mixed null source in conservative boundary conditions [10].

Restrict now to variations tangent to a prescribed-clock, nonexpanding sector,

$$\theta_\xi = 0, \qquad \delta\theta_\xi = 0. \tag{17}$$

Then the scalar source reduces to inaffinity. Restoring the acceleration-valued surface gravity $\kappa = c^2\widehat{\kappa}$ and a physical horizon time τ gives

$$\Theta_\kappa^{(0)} = \frac{c^2}{8\pi G} \int d\tau \int_S \epsilon_S \, \delta\kappa. \tag{18}$$

Using $L_s = c^2/\kappa$,

$$\delta\kappa = -\frac{c^2}{L_s^2} \, \delta L_s, \tag{19}$$

so the same canonical one-form becomes

$$\boxed{\Theta_L^{(0)} = -\int d\tau \int_S \epsilon_S \, \frac{\kappa^2}{8\pi G} \, \delta L_s.} \tag{20}$$

The coefficient of the allowed source variation $-\delta L_s$ is exactly P_L. For a uniform finite cut,

$$F_L = \int_S P_L \, \epsilon_S = A P_L, \qquad \Theta_L^{(0)} = -\int d\tau \, F_L \, \delta L_s. \tag{21}$$

The boundary-potential derivative and the Einstein symplectic potential therefore lead to the same force and pressure by different routes. The first makes the mechanical form transparent; the second establishes that the coefficient is the canonical response to an admissible gravitational boundary-source variation.

The action does not privilege L_s by chain rule alone: any smooth invertible $q = f(\kappa)$ would transform the response. Acceleration length is selected by the independent physical conditions summarized in Sec. 2: it is the minimal length built from c and κ, Rindler fixes its normalization, its conjugate has the form $F = M_\partial \kappa$, it gives force-per-area pressure and $P\,dV$ work, and it becomes the areal radius in Cai–Kim FRW.

3.1. Independent area–boost check

There is also a compact stationary check. Einstein gravity assigns the relative boost angle η of the boundary normals the corner momentum [11, 12]

$$J_\eta = \frac{c^3 A}{8\pi G}. \tag{22}$$

During a stationary interval $\Delta\tau$,

$$\eta = \frac{\kappa\Delta\tau}{c} = \frac{c\Delta\tau}{L_s}. \tag{23}$$

At fixed A and $\Delta\tau$,

$$\boxed{J_\eta \, \delta\eta = -\Delta\tau \, F_L \, \delta L_s.} \tag{24}$$

The same $F_L = A\kappa^2/(8\pi G)$ therefore appears from the independent area–boost corner pair. A force arises because the boost angle is a rate integrated over time while L_s is the reciprocal length representation of that rate.

4. Relation to the standard first law

The usual stationary black-hole first law is not obtained by taking the full differential of $A\kappa$. It is the Hamiltonian variation associated with a specified horizon evolution. Consequently, the absence of an explicit $A\,d\kappa$ term is not an algebraic omission. In ordinary thermodynamic language, $d(TS) = T\,dS + S\,dT$ does not imply that the mass first law must contain $S\,dT$.

Boundary-action treatments make the complementary channel explicit: the two pieces of the surface-Hamiltonian variation become $T\,\delta S$ and $S\,\delta T$ [5, 4]. Since $T \propto 1/L_\kappa$, on the positive branch the second piece is

$$\boxed{S\,\delta T = -F_L \, \delta L_\kappa.} \tag{25}$$

The acceleration-length representation therefore gives the rate-source channel an ordinary mechanical coordinate without adding a second conserved energy to the standard mass first law.

This distinction is clearest for Schwarzschild. The boundary potential is $U_\partial = Mc^2/2$, whereas the mass first law varies $E = Mc^2$. Along the one-parameter Schwarzschild family,

$$dE = T\,dS = 2F_L \, dL_\kappa, \qquad dU_\partial = F_L \, dL_\kappa = \frac{1}{2} dE. \tag{26}$$

The scale work is real, but appending $-P_L dV$ as an independent term to the ordinary mass first law would double-count the constrained Schwarzschild variation.

5. Three physical realizations

5.1. Rindler

For a uniformly accelerated observer in Minkowski spacetime,

$$\kappa = a, \qquad L_\kappa = \frac{c^2}{a}, \qquad P_L^{\mathrm{R}} = \frac{a^2}{8\pi G}. \tag{27}$$

Here L_κ is literally the observer–horizon proper distance on the instantaneous rest slice and the curvature radius of the accelerated worldline. No source mass, center, spacetime curvature, or spherical radius is required. Rindler therefore shows that the pressure is fundamentally a boundary/clock response rather than a formula that depends on a Newtonian source field.

5.2. Schwarzschild

For a Schwarzschild black hole with Killing time normalized at infinity,

$$\kappa = \frac{c^2}{2R_H}, \qquad L_\kappa = 2R_H. \tag{28}$$

With $A_H = 4\pi R_H^2$,

$$\boxed{\begin{aligned} &P_L^{\mathrm{Schw}} = \frac{c^4}{32\pi G R_H^2}, \quad F_L^{\mathrm{Schw}} = \frac{c^4}{8G}, \\ &U_\partial^{\mathrm{Schw}} = \frac{Mc^2}{2}. \end{aligned}} \tag{29}$$

The Smarr relation takes the exact (virial) form

$$\boxed{Mc^2 = TS + F_L L_\kappa, \qquad TS = F_L L_\kappa = \frac{Mc^2}{2}.} \tag{30}$$

The equality is pointwise for every Schwarzschild state; no time averaging is involved. The two products are associated with the area/entropy and normal-scale directions of the same boundary potential. In addition, the pressure agrees in magnitude with the longitudinal Newtonian gravitational field stress $g_N^2/(8\pi G)$ at the horizon because $g_N(R_H) = \kappa$ in this normalization [13]. This is a sectoral correspondence, not an identification of P_L with a covariant bulk stress tensor.

5.3. FRW apparent horizon

For the four-dimensional FRW metric

$$ds^2 = -c^2dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega_2^2\right), \tag{31}$$

the apparent-horizon radius is

$$R_A = \frac{c}{\sqrt{H^2 + kc^2/a^2}}. \tag{32}$$

Let ρ denote total energy density and p_m the bulk material pressure in the same units. The Friedmann constraint and Misner–Sharp energy give

$$E_A = \rho V_A = \frac{c^4 R_A}{2G}, \qquad V_A = \frac{4\pi}{3}R_A^3. \tag{33}$$

Along the homogeneous family of apparent-horizon screens,

$$\boxed{P_{\mathrm{scr}} \equiv \frac{dE_A}{dV_A} = \frac{c^4}{8\pi G R_A^2} = \frac{\rho}{3}.} \tag{34}$$

This is a constrained screen-size response, not the cosmic material pressure p_m and not an unconstrained derivative $-(\partial E/\partial V)_S$.

The instantaneous Cai–Kim prescription sets [14]

$$\kappa_{\mathrm{CK}} = \frac{c^2}{R_A}, \qquad L_{\mathrm{CK}} = R_A, \tag{35}$$

so the general acceleration-length pressure coincides exactly with the screen response,

$$P_L^{\mathrm{CK}} = P_{\mathrm{scr}}. \tag{36}$$

The mechanical differential is especially transparent. From $\Pi_\partial = c^4/(8\pi G R_A)$, $A_A = 4\pi R_A^2$, and Eq. (34),

$$\boxed{dE_A = \Pi_\partial\, dA_A - P_{\mathrm{scr}}\, dV_A.} \tag{37}$$

Only afterward, using $\Pi_\partial dA_A = T_{\mathrm{CK}} dS_A$, this becomes

$$\boxed{dE_A = T_{\mathrm{CK}}\, dS_A - P_{\mathrm{scr}}\, dV_A.} \tag{38}$$

This is a screen-state law. Cai and Kim's original relation instead tracks matter energy crossing a momentarily fixed apparent horizon, $\delta Q_{\mathrm{matt}} = T_{\mathrm{CK}} dS_A$ [14]. The flux law sorts change by carrier; Eq. (38) resolves the changing screen state by boundary coordinate.

The pressure also gives a concise form of the Friedmann geometry. With the Einstein coupling $\kappa_{\mathrm{E}} = 8\pi G/c^4$,

$$\boxed{K_{S^2} \equiv \frac{1}{R_A^2} = \kappa_{\mathrm{E}} P_{\mathrm{scr}},} \tag{39}$$

where K_{S^2} is the Gaussian curvature of the round apparent-horizon sphere. The acceleration equation becomes

$$\boxed{{}^{(2)}\mathcal{R}_\perp \equiv \frac{2\ddot{a}}{ac^2} = -\kappa_{\mathrm{E}}(P_{\mathrm{scr}} + p_m),} \tag{40}$$

where ${}^{(2)}\mathcal{R}_\perp$ is the Ricci scalar of the two-dimensional time–radial geometry normal to the symmetry spheres, with the curvature-sign convention used here. The positive screen pressure sets the transverse horizon-curvature scale; its sum with bulk pressure controls acceleration or deceleration. Accelerated expansion begins at $p_m < -P_{\rm scr}$.

Full Kodama–Hayward surface gravity measures a distinct normal cross-focusing response [16, 17]:

$$\boxed{\frac{\kappa_H^s}{\kappa_{\rm CK}} = \frac{1}{4}\left(\frac{p_m}{P_{\rm scr}} - 1\right).} \tag{41}$$

At the trace-free radiation point $p_m = P_{\rm scr}$, the Hayward rate vanishes while the Cai–Kim screen pressure remains finite. The two prescriptions answer different questions and should not be identified.

A recent Annals of Physics study by Kong independently emphasized the constrained apparent-horizon energy–volume slope by defining $P_{\rm eff} \equiv -dE_A/dV_A$ [18]. In four-dimensional Einstein FRW this gives $P_{\rm eff} = -P_{\rm scr} = -\rho/3$. The present construction differs in two respects. First, the sign of $P_{\rm scr}$ is fixed by the boundary work convention $dE_A = T_{\rm CK} dS_A - P_{\rm scr} dV_A$. Second, $P_{\rm scr}$ is not introduced as an FRW-only effective definition: it is the Cai–Kim realization of the general acceleration-length pressure already obtained from the Einstein boundary source response and realized independently in Rindler and Schwarzschild.

6. Thermal translation and entropy density

Thermality is not used to derive the pressure. Where a regular Hawking–Unruh/KMS state exists, Euclidean continuation of the same normalized normal flow fixes

$$c\Delta\tau_E = 2\pi L_\kappa, \qquad k_B T = \frac{\hbar c}{2\pi L_\kappa} = \frac{\hbar|\kappa|}{2\pi c}. \tag{42}$$

On the positive thermodynamic branch, Einstein entropy gives $U_\partial = TS$ and $\Pi_\partial dA = T\,dS$, so Eq. (10) becomes

$$\boxed{dU_\partial = T\,dS - F_L\,dL_\kappa = T\,dS - P_L\,dV_\perp.} \tag{43}$$

The same positive scale therefore carries the complementary laws

$$\boxed{P_L(L_\kappa) = \frac{c^4}{8\pi G L_\kappa^2}, \qquad k_B T(L_\kappa) = \frac{\hbar c}{2\pi L_\kappa}.} \tag{44}$$

Their ratio reproduces the Einstein entropy density,

$$\boxed{\frac{k_B T}{P_L L_\kappa} = 4\ell_{\rm P}^2, \qquad \frac{S}{A} = \frac{P_L L_\kappa}{T} = \frac{k_B}{4\ell_{\rm P}^2}.} \tag{45}$$

Here pressure across acceleration length is the mechanical boundary-energy density, while temperature provides its thermal conversion. The factor four is the ratio of the Einstein boundary normalization $1/(8\pi G)$ to the Hawking–Unruh normalization $1/(2\pi)$; the surface-gravity scale cancels.

7. Relation to prior work and scope

The ingredients supporting the construction are established: the area–surface-gravity structure of black-hole mechanics [1, 2]; membrane surface pressure/tension [3]; the $A\kappa$ Kodama–Noether and boundary Hamiltonian structures [4, 5]; the null mixed scalar source [8, 9, 10]; Rindler/Unruh normalization [7]; Hayward's unified first law [16]; and the Cai–Kim apparent-horizon construction [14]. The present contribution is not the existence of those ingredients individually. It is the identification of acceleration length as the normal mechanical source coordinate that simultaneously gives the canonical response $P_L = \kappa^2/(8\pi G)$, the finite force $F_L = AP_L$, pressure–volume work, and one consistent Rindler–Schwarzschild–FRW dictionary.

The distinction between an invertible coordinate change and a physical source choice is central. Algebra alone permits infinitely many $q = f(\kappa)$. The boundary action establishes that the normal rate is genuine source data; the physical requirements of a length coordinate, exact Rindler calibration, $F = M_\partial \kappa$, force-over-area pressure, $P\,dV$ work, thermal reciprocal scale, and the Cai–Kim radius then select c^2/κ with unit normalization. The stationary area–boost derivation supplies an independent canonical check of the same force.

The result is conditional on a prescribed physical horizon flow and clock. At $\kappa = 0$, the reciprocal chart moves to infinity and the regular parent rate must be used. On a generic expanding null boundary the scalar source is $\mu = \widehat{\kappa} + \theta/2$, not surface gravity alone. Generic moving horizons therefore require their own normal-flow prescription; the Cai–Kim equilibrium calibration and the full Hayward cross-focusing rate are deliberately kept distinct.

The construction does not introduce a new bulk stress tensor, alter Einstein's equation, or claim an additional independently conserved horizon energy. It identifies a normal mechanical response of established gravitational boundary data and shows how that response is realized in three standard horizon sectors.

8. Conclusion

The Einstein boundary potential has a familiar transverse area direction and a normal surface-gravity direction. Writing the latter through the signed acceleration length $L_s = c^2/\kappa$ gives

$$\boxed{\begin{aligned} &dU_\partial = \Pi_\partial\, dA - F_L\, dL_s, \\ &F_L = AP_L, \qquad\qquad P_L = \frac{\kappa^2}{8\pi G}. \end{aligned}} \tag{46}$$

At fixed area this is force-displacement and pressure–volume work. The Einstein null-boundary symplectic potential independently places the same coefficient beside the allowed acceleration-length source variation, and the stationary area–boost pair gives the same force. The central mechanical interpretation therefore rests on both an energy derivative and canonical gravitational boundary structure.

Rindler makes the reciprocal scale a literal proper distance; Schwarzschild gives $L_\kappa = 2R_H$ and an exact $TS = F_L L_\kappa = Mc^2/2$ Smarr split; Cai–Kim FRW makes $L_\kappa = R_A$ and gives $P_{\rm scr} = \rho/3$ together with $dE_A = T_{\rm CK} dS_A - P_{\rm scr} dV_A$. Hawking–Unruh thermality enters only afterward and identifies the same scale as the reduced thermal length. The resulting pressure and temperature laws reproduce the quarter-area entropy density while retaining distinct classical-mechanical and semiclassical-thermal meanings.

The main result is consequently simple but not merely algebraic: surface gravity is the regular normal rate, acceleration length is its physically selected mechanical coordinate, and the Einstein boundary response to that coordinate is a genuine normal force and pressure.

Declaration of generative AI and AI-assisted technologies

During the preparation of this work the author used OpenAI GPT in order to identify relevant literature and assist with final presentation. All physical insight and results are from the author. After using this tool/service, the author reviewed and edited the content as needed and takes full responsibility for the content of the published article.

Data availability

No data were created or analyzed in this theoretical study.

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